Locally Robust Inference for Non-Gaussian Linear Simultaneous Equations Models[*]

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Abstract

All parameters in linear simultaneous equations models can be identified (up to permutation and sign) if the underlying structural shocks are independent and at most one of them is Gaussian. Unfortunately, existing inference methods that exploit such identifying assumptions suffer from size distortions when the true distributions of the shocks are close to Gaussian. To address this *weak non-Gaussian* problem we develop a locally robust semi-parametric inference method which is simple to implement, improves coverage and retains good power properties. The finite sample properties of the methodology are illustrated in a large simulation study and an empirical study for the returns to schooling.

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1 Introduction

The linear simultaneous equations model (LSEM) is a benchmark model used to analyze general equilibrium relationships in economics. It was placed in its modern form by Haavelmo (1943, 1944), building on Frisch (1933) and Tinbergen (1939) among others. As is well known, without additional restrictions, not all parameters of the LSEM can be uniquely identified from the first and second moments of the observed data series, see Dhrymes (1994) for an in-depth discussion.

Interestingly, this identification problem vanishes (up to permutation and scale) when the underlying structural shocks are independent and at most one of them follows a Gaussian distribution (e.g. Comon, 1994). This identification approach has a long history in the statistics and signal processing literatures where it is often referred to as independent components analysis, see Hyvärinen, Karhunen and Oja (2001) for a textbook treatment. More recently, this approach has been adopted in the econometrics literature, where interest has centered on developing methodology for conducting inference on the parameters of various LSEMs based on non-Gaussian identification (e.g. Gouriéroux, Monfort and Renne, 2017).

Unfortunately, if in the true data generating process multiple structural shocks follow a Gaussian distribution some structural parameters may be under- or un-identified and standard inference methods that aim to exploit non-Gaussian distributions may fail to control size. Moreover, as is typical in models with points of identification failure, such behavior is also observed if the true distributions of the shocks are sufficiently close to Gaussianity. Intuitively, in such *weakly non-Gaussian* settings the available identifying information is limited relative to sampling variation leading to asymptotic coverage distortions when using standard inference methods, such as maximum likelihood and moment condition based methods.

Similar (weak) identification problems occur in many other econometric models, e.g. instrumental variable models, nonlinear regression models and many others, see Staiger and Stock (1997), Stock and Wright (2000) and Andrews and Mikusheva (2015) for some examples. The key difference between this existing literature and the non-Gaussian LSEM is that, in the latter, the parameters responsible for the possible identification failure are density functions, i.e. infinite dimensional parameters. Therefore, whilst conceptually the identification problem is the same, providing robust inferential methods requires a new approach which is capable of handling identification failure caused by infinite dimensional nuisance parameters.

To this extent, this paper develops a new approach for conducting inference in LSEMs that is inspired by the weak identification robust methods developed in econometrics (e.g.

Stock and Wright, 2000; Kleibergen, 2005; Andrews and Mikusheva, 2015) and the general semiparametric statistical theory that is discussed in Bickel et al. (1998) and van der Vaart (2002). In brief, we treat the LSEM as a semiparametric model, where the densities of the independent structural shocks are treated non-parametrically, and we construct confidence bands for the possibly unidentified structural parameters of interest by inverting semiparametric score tests. The approach efficiently exploits non-Gaussianity when it is present in the data and yields confidence bands which do not asymptotically under-cover under sequences of densities that are local (in a $\sqrt{n}$ neighborhood) to the true density. Moreover, the test is easy to implement and the critical values accompanying the test statistic are standard chi-squared.

The effective score test that we propose is the semi-parametric analog of the Neyman-Rao test (e.g. Neyman, 1979; Hall and Mathiason, 1990). In the conventional Neyman-Rao test the scores for the parameter of interest are orthogonalized with respect to the scores for the *finite dimensional* nuisance parameters. In our setting the nuisance parameter includes the densities of the shocks, i.e. an *infinite dimensional* parameter. While such nuisance functions result in the orthogonal projection being more technically demanding to derive, the main idea of Neyman (1979) continues to apply.

Formally, we show that the semi-parametric score test is locally robust in the sense that its null rejection probability is no greater than the nominal level under parameter sequences that can be described by local deviations from the true parameters which satisfy the null hypothesis. In particular, the null rejection probability of the test is controlled for sequences of densities that converge at a $\sqrt{n}$ rate to the Gaussian density, a point of identification failure. These sequences are the natural counterpart in our setting to the "weak identification asymptotics" as found in, for example, Staiger and Stock (1997); Stock and Wright (2000); Moreira (2003); Kleibergen (2005); Andrews and Mikusheva (2015). Moreover, they are those considered in the theory of Kaji (2021) who studies estimation in weakly identified semiparametric models.[1] In addition, we show that under strong identification, which requires all errors to be (sufficiently) non-Gaussian, the score test is semi-parametrically efficient in the sense that it attains various local asymptotic power bounds for testing scalar or vector valued parameters (cf. Choi, Hall and Schick, 1996).

We evaluate the finite sample performance of the semiparametric score test in a large simulation study. We find that the null rejection probability of our test remains close to the nominal level for all distributions considered, including those which are "close" to the Gaussian distribution and the Gaussian distribution itself. In contrast, tests that are based on the

[1] See also Andrews and Mikusheva (2022) who study weakly identified GMM models using the same type of local sequences.

sampling variation of (pseudo)-maximum likelihood or GMM estimators often substantially over-reject in weakly non-Gaussian settings. Further, for moderate sample sizes the power of the semiparametric test is comparable to the parametric score test that relies on knowing the functional form of the density. When the parametric density of the (pseudo)-maximum likelihood score test is misspecified the semi-parametric test is always found to be preferable.

To showcase the empirical value of our methodology we adopt the score test to construct confidence bands for the effect of education on wages. To do so, we consider a special case of the LSEM model: the linear instrumental variable (IV) model. We show that the presence of independent non-Gaussian errors allows to (i) strengthen identification for the case where the instrument is assumed exogenous and (ii) test and correct for endogenous instrumental variables. We emphasize that our theory allows for, and is locally robust to, weak instruments.

For the model specification and data considered in Card (1995) we find that inverting the semi-parametric score test gives the shortest confidence intervals for the returns to education which are, for instance, shorter when compared to confidence intervals based on the Anderson and Rubin (1949) statistic. Also, when we relax the instrument exogeneity assumption and use non-Gaussianity to identify the returns to education, we find that (i) the assumption that the proximity to college instrument is exogenous cannot be rejected and (ii) the confidence interval for the returns to education remains precisely estimated. In contrast, using alternative but non-efficient methods we find considerably larger confidence sets when relaxing the instrument exogeneity assumption.

In general, this paper highlights the problem of weak non-Gaussianity and provides a solution in the setting of i.i.d. linear simultaneous equations models. We point out that similar non-Gaussian identification approaches have been adopted in other settings and it is likely that weak non-Gaussianity continues to cause inference problems for standard MLE and GMM methods in these settings. Prominent examples include (i) structural VAR(MA) models (Lanne and Lütkepohl, 2010; Moneta et al., 2013; Lanne, Meitz and Saikkonen, 2017; Maxand, 2018; Gouriéroux, Monfort and Renne, 2019; Tank, Fox and Shojaie, 2019; Herwartz, 2019; Herwartz, Lange and Maxand, 2019; Bekaert, Engstrom and Ermolov, 2020, 2021; Lanne and Luoto, 2021; Guay, 2021; Sims, 2021; Moneta and Pallante, 2022; Fiorentini and Sentana, 2023; Velasco, 2022; Davis and Ng, 2022; Drautzburg and Wright, 2023), (ii) measurement error models (e.g. Reiersøl, 1950; Kapteyn and Wansbeek, 1983; Dagenais and Dagenais, 1997; Erickson and Whited, 2000, 2002; Bonhomme and Robin, 2009), and (iii) triangular systems (e.g. Lewbel, Schennach and Zhang, 2023). In future work we aim to extend our semi-parametric inference approach to cover these more general settings. The supplementary material that accompanies this paper provides a step in this direction by considering a class of nonlinear simultaneous equations models.

Further, as mentioned above, this paper shows that the proposed semi-parametric score test has null rejection probability asymptotically bounded by the nominal level under weak identification asymptotics, i.e. under parameter sequences representing local deviations from the true parameters (which satisfy the null hypothesis).[2] A global uniformity statement — not restricted to local sequences— as developed in, inter alia, Andrews and Cheng (2012), Andrews and Cheng (2013) and Andrews, Cheng and Guggenberger (2020) for models where identification failure is characterized by a finite dimensional parameter, is beyond the scope of this paper. It remains an open question whether global uniformity can be achieved in a meaningful way, i.e. without unreasonably restricting the parameter space and/or equipping it with a very strong metric, in models where identification failures are characterized by infinite dimensional parameters.

The remainder of this paper is organized as follows. In the next section we provide a simple example that illustrates the identification problem and intuitively discusses our solution. Section 3 presents the main LSEM model and provides the implementation details for the effective score test. Section 4 discusses the main theoretical results including the required assumptions. Sections 5 and 6 summarize the results from the simulation and empirical studies. Section 7 concludes. Unless otherwise mentioned all proofs are provided in the Appendix. Any references to sections, equations, lemmas etc. which start with "S" refer to the supplementary material.

2 Illustrative example

In this section we use a simple example to illustrate: (i) the identification problem in LSEMs, (ii) why conventional inference methods suffer from size distortions when the structural shocks have densities close to Gaussian and (iii) how our proposed approach aims to circumvent such distortions.

The identification problem

Consider the simple bi-variate model

$$Y_i = A^{-1}\epsilon_i \ , \qquad i = 1, \ldots, n \ , \tag{1}$$

[2] Such sequences have also been used to model weak identification in semi-parametric models in Kaji (2021); Andrews and Mikusheva (2022).

where Y_i is a vector of observable variables, A is a rotation matrix (i.e. $A^{-1} = A'$ and $\det(A) = 1$) and ϵ_i is a vector with independent structural shocks ϵ_{ik}, for $k = 1, 2$, that have mean zero, unit variance and common density η. For concreteness, we will parameterize the rotation matrix as follows

$$A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} , \tag{2}$$

where $\alpha \in [0, 2\pi)$ and we let α_0 denote the true parameter.[3]

Model (1) has two parameters: the parameter of interest α and the infinite dimensional nuisance parameter η. Suppose for now that η is known and let the log likelihood function for Y_i be denoted by $\ell_\alpha(\cdot)$. The parameter α is locally identified if the expected score of $\ell_\alpha(Y_i)$ with respect to α is non-zero for all $\alpha \neq \alpha_0$ in a neighborhood of α_0.

Whether local identification occurs depends crucially on η. To illustrate, consider the case where η is equal to the Gaussian density. Since ϵ_i is normalized we have

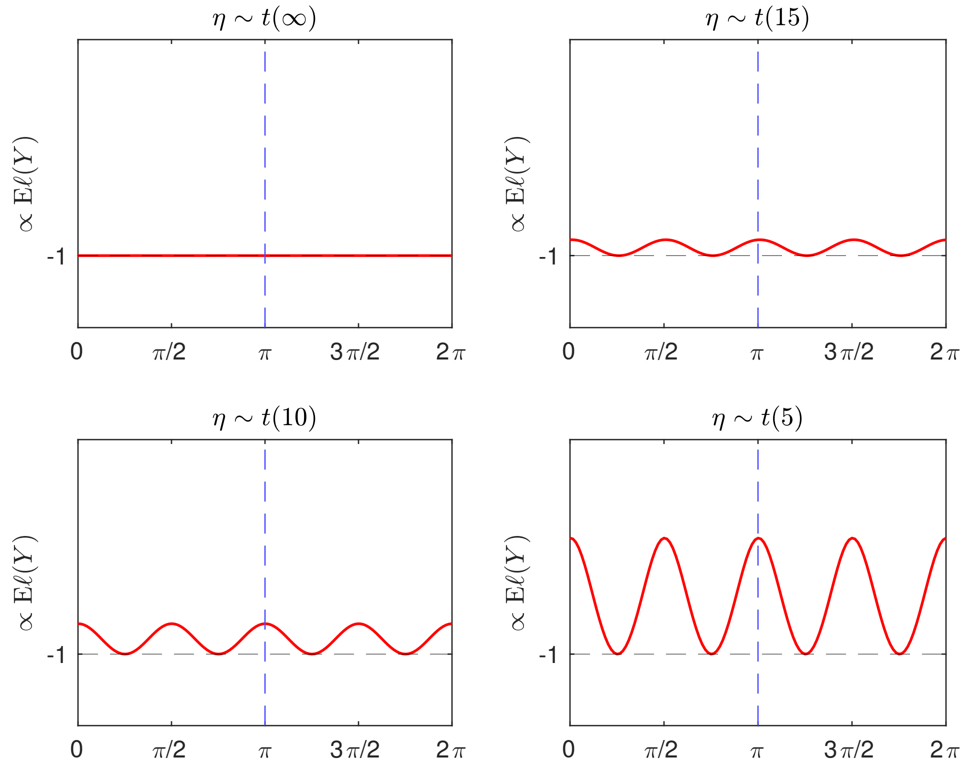
$$\mathbb{E}\ell_\alpha(Y_i) \propto -\frac{1}{2}\mathbb{E}(AY_i)'(AY_i) = -1 ,$$

and hence the expected loglikelihood takes the same value irrespective of α. This is plotted in the top left panel of Figure 1, where we show the expected likelihood $\mathbb{E}\ell_\alpha(Y_i)$ as a function of α with $\alpha_0 = \pi$ as the true parameter (an arbitrary choice). This illustrates the standard identification problem in linear simultaneous equations models: without additional identifying restrictions, the impact effects of the structural shocks are not identifiable when the structural shocks follow a Gaussian distribution.

The other plots in Figure 1 show that this is no longer the case when we move away from the Gaussian distribution. In each case the expected gradient becomes non-zero at values $\alpha \neq \alpha_0$ in a neighbourhood of α_0, i.e. local identification occurs. While for the (standardized) Student's t distribution with five degrees of freedom (i.e. $t(5)$) the change in the value of the expected likelihood is substantial it is easy to see that for more modest deviations from Gaussianity (e.g. $t(15)$) the difference is less pronounced. Further, note that non-Gaussian densities do not ensure α is globally identified, instead identification is only up to permutation and sign of the shocks.

[3] Note that in our general framework we will not restrict A to be a rotation matrix nor η to be common. This example is chosen for exposition purposes only and corresponds to the case where the variance of Y_i is normalized to unity.

Figure 1: (Weak) Non-Gaussian Identification



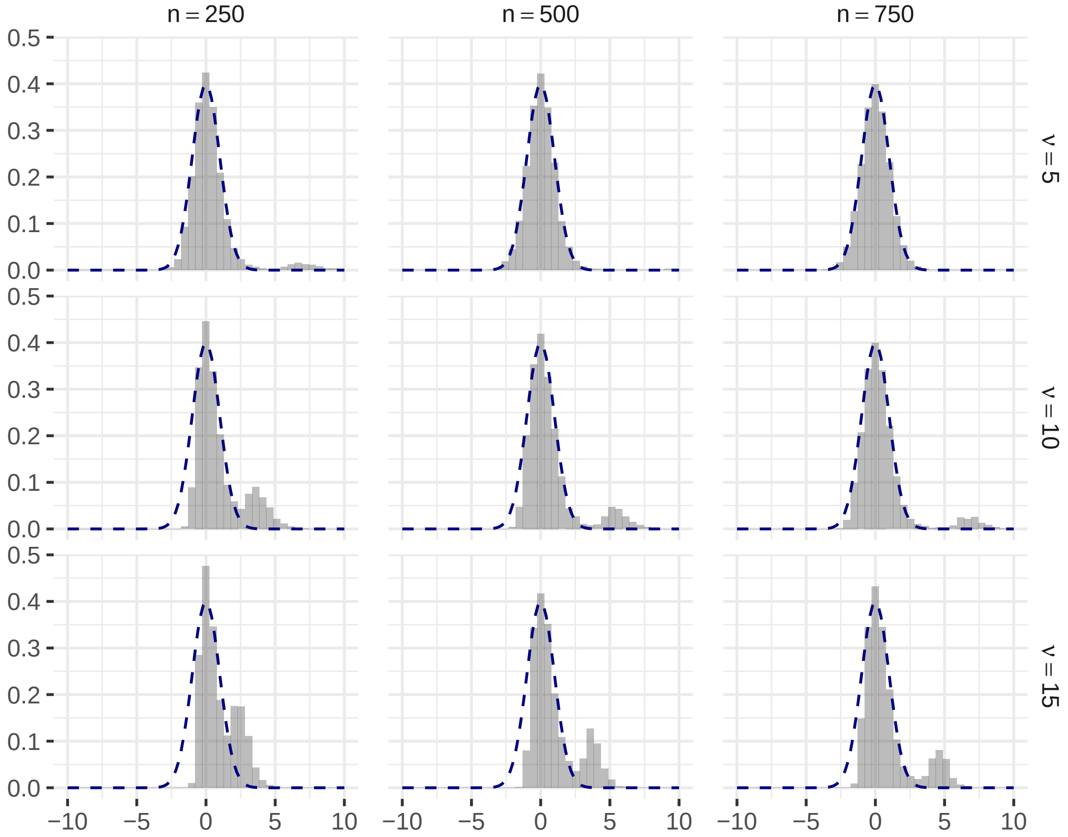
Notes: In the figure we show the expected log likelihood (red line) as a function of $\alpha \in [0, 2\pi)$. The true value is $\alpha_0 = \pi$.

Finite sample size distortions

In population α is always locally identified when all but one component of η is non-Gaussian (e.g. Comon, 1994, Theorem 11), but this is not sufficient for good performance of standard testing procedures in finite samples. In particular, if the structural shocks are too close to Gaussian, the available identifying information may be small relative to the sampling variability. Standard asymptotic approximations are not reliable in this setting and, as a result, testing procedures based on these approximations may fail to provide reliable inference.

To illustrate how the density η affects standard inference methods in finite sample, we draw 5000 samples $\{Y_i\}_{i=1}^n$ from model (1) for different η's using different sample sizes $n = 250, 500, 750$. Figure 2 shows the finite sample distribution of the t-statistic for the hypothesis $H_0 : \alpha = \alpha_0$, with $\alpha_0 = \pi$, based on the maximum likelihood estimator under the assumption that η is known. The blue dashed lines show the $\mathcal{N}(0, 1)$ density that corresponds to the usual

Figure 2: Poor asymptotic approximation close to Gaussianity



Notes: In the figure we show the finite sample distribution of the t-statistic based on the maximum likelihood estimator of α (the true value is $\alpha_0 = \pi$) for different sample sizes (n) and different degrees of freedom (ν) in the (standardised) t distribution, all based on 5000 replications. Letting $\hat{\alpha}$ be the ML estimator and α_0 the the null hypothesis value of α, the t-statistic used is $t = \sqrt{n}(\hat{\alpha} - \alpha_0) \times \sqrt{\hat{I}}$, with $\hat{I}$ the usual outer product of gradients (OPG) estimator of the (Fisher) information: $\hat{I} = \frac{1}{n} \sum_{i=1}^{n} \dot{\ell}_{\hat{\alpha}}(Y_i)^2$, with $\dot{\ell}_\alpha = \nabla_\alpha \ell_\alpha$.

limit of the t-statistic. As can clearly be seen in this figure, the quality of the approximation provided by the standard Gaussian depends crucially on the underlying density, η. For a given sample size, the approximation deteriorates substantially the closer η is to a standard Gaussian density.

This deterioration results in poor size control of standard tests. Table 1 shows the empirical rejection frequencies for three standard tests in the same setting: Wald (W), likelihood ratio (LR) and Lagrange multiplier (LM) (or score) tests, all computed under the assumption that η is known. The empirical rejection frequencies correspond to the test for $H_0 : \alpha = \alpha_0$ with nominal level $a = 0.05$, where the critical values are based on the standard χ_1^2 asymptotic approximation.

We find that the Wald test is severely size distorted for η close to Gaussian; in view of the poor quality of asymptotic approximation depicted in Figure 2 this is not surprising. As η gets closer to Gaussianity, the likelihood ratio test starts to under-reject as when α is poorly identified the likelihood values are very similar. Both of these tests are based on estimates of α and, in weakly identified settings, such estimates will be inaccurate.

In contrast, the score test (LM) shows correct size as it fixes $\alpha = \alpha_0$ under the null and α does not need to be (well) identified for this test to be correctly sized. Intuitively, with α fixed and η known there are no further unknown elements in the scores and the remaining uncertainty is due to sampling variation. This observation provides the first building block for the test we will construct: it will be a score type test which fixes $\alpha = \alpha_0$ under the null.

Table 1: Rejection Frequencies for ML tests close to Gaussianity

	t(15)			t(10)			t(5)		
n	W	LM	LR	W	LM	LR	W	LM	LR
250	25.26	4.42	3.74	20.56	4.24	4.04	8.88	4.84	4.08
500	21.76	4.54	4.52	13.10	4.38	3.60	6.38	4.42	4.92
750	17.12	4.96	3.94	9.90	4.88	3.42	6.12	5.28	5.64

Notes: The table shows the empirical rejection frequencies for the three maximum likelihood tests, under the assumption that η is known and based on 5000 Monte Carlo replications for the baseline model $Y_i = R'\epsilon_i$. The test has nominal level $a = 0.05$.

Towards a semi-parametric score test

In practice, η will be unknown. To build up to our semi-parametric approach, consider first the case where η is known up to a finite dimensional parameter vector, say ν. For example ν may include the degrees of freedom of the Student's t distribution.

For such cases Neyman (1979) proposed a convenient extension of the standard score test, that amounts to first orthogonalizing the scores for α with respect to the scores for ν and then computing a quadratic form of the score statistic. To illustrate let $\dot{\ell}(Y_i) = (\dot{\ell}_\alpha(Y_i), \dot{\ell}_\nu(Y_i))'$, $\dot{\ell}_\alpha(Y_i) = \nabla_\alpha \ell_{\alpha,\nu}(Y_i)$, $\dot{\ell}_\nu(Y_i) = \nabla_\beta \ell_{\alpha,\nu}(Y_i)$ and $\hat{I} = \frac{1}{n}\sum_{i=1}^n \dot{\ell}(Y_i)\dot{\ell}(Y_i)'$, denote the score and information matrix for α and ν. The Neyman-Rao score test statistic is given by

$$S = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \hat{\kappa}(Y_i)\right)' \hat{\mathcal{I}}^{-1} \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n \hat{\kappa}(Y_i)\right) \ ,$$

with

$$\hat{\kappa}(Y_i) = \dot{\ell}_\alpha(Y_i) - \hat{I}_{\alpha\nu}\hat{I}_{\nu\nu}^{-1}\dot{\ell}_\nu(Y_i) \qquad \text{and} \qquad \hat{\mathcal{I}} = \hat{I}_{\alpha\alpha} - \hat{I}_{\alpha\nu}\hat{I}_{\nu\nu}^{-1}\hat{I}_{\nu\alpha} \ ,$$

where $\hat{I}_{\cdot\cdot}$ denote the corresponding blocks of $\hat{I}$.[4] The (estimated) orthogonalized scores $\hat{\kappa}(\cdot)$ are often referred to as the (estimates of the) effective scores and $\hat{\mathcal{I}}$ is the corresponding (estimate of the) effective information matrix.

This score statistic is usually evaluated as $\alpha = \alpha_0$ and some $\sqrt{n}$ consistent estimate for ν. Whenever such an estimate exists, S will converge to a standard χ^2 limit under the null provided that $\hat{\mathcal{I}}$ is invertible.[5] In such cases, tests based on S retain correct size regardless of whether or not α is well identified making them attractive for settings where identification failure due to finite dimensional nuisance parameters is a concern (e.g. Andrews and Mikusheva, 2015).

Unfortunately, there are two distinct problems that may arise in the solution sketched above. First and most practically relevant, modeling the deviations from the Gaussian density in a parametric manner may result in biases and/or lower power whenever the true density lies outside of the parametric class considered. Second, parametric deviations from the Gaussian density as captured by ν generally nest the Gaussian distribution. In many such cases the information matrix associated to ν, i.e. $I_{\nu\nu}$, becomes singular when the true density is Gaussian. Sometimes this problem can be circumvented by re-parametrizing ν, e.g. parameterize $\tilde{\nu} = \nu^{-1}$ for the degrees of freedom of the Student's t or for a skewed-normal

[4] This is numerically equivalent to the "usual" score test when the nuisance parameter ν is estimated by (restricted) maximum likelihood under the null hypothesis (Kocherlakota and Kocherlakota, 1991).

[5] In our results below we allow $\hat{\mathcal{I}}$ to be singular and rely on an eigenvalue truncated generalized inverse, see also Andrews (1987), Lütkepohl and Burda (1997) and Andrews and Guggenberger (2019).

one can adopt the centered parametrization of Azzalini and Capitanio (2014, Section 3.1.4). However, for other examples, such as mixtures of normals, there are no available transformations that prevent the information matrix from becoming singular under Gaussianity. That is, ν itself becomes unidentified (Rothenberg, 1971, Theorem 1) and consistent estimators of ν do not exist.

We note that these problems interact as solving the identification problem for ν can be done by adopting a pseudo maximum likelihood approach that fixes ν at some reasonable value (e.g. Gouriéroux, Monfort and Renne, 2017), but this immediately implies that the true likelihood may be far away from the fixed pseudo likelihood, resulting in a test with little power.

In the present paper, we do not assume that the parametric form of η is known up to a finite dimensional parameter vector but instead treat η non-parametrically. To avoid the creation of additional identification problems we rely on B-spline estimators to non-parametrically estimate the aspect of η which is necessary to implement our procedure: the log density score of η (i.e. the logarithmic derivative of η). Unlike the finite dimensional parameters ν discussed above, the log density score does not suffer from identification problems at Gaussianity.

Despite such changes, the underlying logic of our approach is similar to that sketched above. We first orthogonalize the score for α with respect to the scores for η and obtain a semi-parametric analog of the conventional Neyman-Rao score test. This requires technical adjustments as the scores with respect to η need to be defined differently and the projection with respect to η scores requires more care. For this we follow the semi-parametric literature as outlined in Bickel et al. (1998) and van der Vaart (2002).

3 Locally robust inference for LSEMs

In this section we propose a semi-parametric score test for testing parameters in a general class of linear simultaneous equations models. We first introduce the model class and give some motivating examples. Thereafter, we present a heuristic derivation for the score test and the exact implementation details. All theoretical properties including the main assumptions are deferred to the next section.

3.1 General model, objectives and examples

We consider the linear simultaneous equations model for a random sample of K endogenous variables Y_i, d exogenous variables $X_i = (1, \tilde{X}_i')'$ and K independent structural shocks ϵ_i, which have mean zero and unit variance. Specifically, we have

$$Y_i = BX_i + A^{-1}\epsilon_i \ , \qquad i = 1, \dots, n \ , \tag{3}$$

where we observe $W_i = (Y_i', X_i')'$ and the matrices B and A^{-1} map the explanatory variables and the structural shocks to the endogenous variables. The density functions of the components of $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{iK})'$ are denoted by $(\eta_1, \dots, \eta_K)$ and the density of $\tilde{X}_i$ is given by η_0. We set $\eta = (\eta_0, \eta_1, \dots, \eta_K)$.

As illustrated in the previous section, depending on the shapes of $\eta_1, \dots, \eta_K$ we may not be able to identify all parameters in A. To model this we let $A = A(\alpha, \sigma)$, where $A(\alpha, \sigma)$ is a function of both (i) the parameters α which may suffer from identification failure and (ii) the well-identified parameters in σ. We let $\alpha \in \mathcal{A} \subset \mathbb{R}^{L_\alpha}$ and set $\beta = (b, \sigma) \in \mathcal{B} \subset \mathbb{R}^{L_b} \times \mathbb{R}^{L_\sigma} = \mathbb{R}^{L_\beta}$, with $b = \text{vec}(B)$.

In this paper we leave the parametrization of $A = A(\alpha, \sigma)$ largely unspecified. In Assumption 1 we state the formal requirements and subsequently provide examples that can be adopted within our general framework. We stress that the dimensions of α and σ are fixed, as is the dimension of Y_i. As such our framework does not deal with high dimensional LSEMs. A special case of model (3) is obtained when setting $B = 0$ for which the model reduces to the baseline model for independent components analysis (e.g. Hyvärinen, Karhunen and Oja, 2001). Further, after pre-whitening the residuals we obtain the model (1) from the illustrative example.

The general LSEM (3) depends on the following parameters

$$\theta = (\gamma, \eta) \ , \quad \text{with} \quad \gamma = (\alpha, \beta) \quad \text{and} \quad \beta = (b, \sigma) \ , \tag{4}$$

where $\gamma \in \Gamma = \mathcal{A} \times \mathcal{B}$ summarizes all finite dimensional parameters, including the possibly weakly identified α and the well identified β, and η includes the infinite dimensional parameters, i.e. the densities of the shocks for which the parameter space will be formalized below.

We are interested in testing the possibly weakly identified parameters α. To do so, we consider the hypothesis

$$H_0 : \alpha = \alpha_0 \qquad \text{against} \qquad H_1 : \alpha \neq \alpha_0 \ . \tag{5}$$

Tests for such H_0 can then be inverted to yield confidence sets for α. A related set-up is found in Risk, Matteson and Ruppert (2019) and Jin, Risk and Matteson (2019) who assume that the structural shocks can be separated into *exactly* Gaussian and non-Gaussian shocks.

We do not impose such structure, but we note that if indeed shocks can be separated in this way our approach will remain valid, but likely less efficient when compared to Risk, Matteson and Ruppert (2019).

Parameterizing the LSEM In practice, we can adopt different parametrizations for modeling $A = A(\alpha, \sigma)$ in (3). A general requirement is that A is non-singular and that it is sufficiently smooth with respect to α and σ. The following assumption formalizes these conditions.

Assumption 1. *Define the partial derivative matrices $D_{\alpha,l} = \partial A(\alpha,\sigma)/\partial\alpha_l$, for $l = 1, \ldots, L_\alpha$, and $D_{\sigma,m} = \partial A(\alpha,\sigma)/\partial\sigma_m$, for $m = 1, \ldots, L_\sigma$. Further, for each $i, j \in \{1, \ldots, K\}$, $l \in \{1, \ldots, L_\alpha\}$ and $m \in \{1, \ldots, L_\sigma\}$ define $\zeta^\alpha_{l,k,j} := [D_{\alpha,l}]_{k\bullet} A^{-1}_{\bullet j}$ and $\zeta^\sigma_{m,k,j} := [D_{\sigma,m}]_{k\bullet} A^{-1}_{\bullet j}$, where the notation $M_{\bullet j}$ or $M_{j\bullet}$ denotes the jth column or row (respectively) of a matrix M. We assume that for all $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$*

1. *$A(\alpha, \sigma)$ is non-singular*

2. *$(\alpha, \sigma) \to A(\alpha, \sigma)$ is continuously differentiable*

3. *$(\alpha, \sigma) \to \zeta^\alpha_{l,k,j}(\alpha, \sigma)$ and $(\alpha, \sigma) \to \zeta^\sigma_{m,k,j}(\alpha, \sigma)$ are locally Lipschitz continuous for all j, k, l, m*

The following examples illustrate some possible parametrizations that are of practical interest and satisfy the smoothness assumptions.

Example 1 (Supply and Demand)**.** *Following Working (1927)'s canonical analysis of supply and demand curves let Y^s_{i1} and Y^d_{i1} denote the quantity demanded and supplied of some good with price Y_{i2}. In equilibrium we have $Y^d_{i1} = Y^s_{i1}$ and a simple model (omitting covariates for convenience) is given by*

$$\begin{aligned} Y_{i1} &= \alpha_1 Y_{i2} + \sigma_1 \epsilon_{i1} \qquad \text{(demand)} \\ Y_{i1} &= \sigma_3 Y_{i2} + \sigma_2 \epsilon_{i2} \qquad \text{(supply)} \end{aligned}$$

where ϵ_{i1} and ϵ_{i2} are independent demand and supply shocks. We can accommodate this model in our general framework by letting $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and defining the mapping $A(\alpha, \sigma)$ according to

$$A(\alpha, \sigma) = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\alpha_1 \\ 1 & -\sigma_3 \end{bmatrix} .$$

Note that even with non-Gaussian errors, which we do not assume, the matrix $A(\alpha, \sigma)$ is only identifiable up to post multiplication by DP, where P is a permutation matrix and D a *diagonal matrix with elements* ±1 *on the main diagonal (e.g. Comon, 1994, Theorem 11). In applications we could impose sign restrictions to select the permutation that is of economic interest. For instance, here we could impose* $\alpha_1 \leq 0$ *and* $\sigma_3 \geq 0$ *to ensure that the demand curve is downward sloping and the supply curve is upward sloping, as well as* $\sigma_1, \sigma_2 > 0$ *to ensure that the scales are positive. As such we would only test values for* α_1 *in (5) that satisfy the sign restrictions.*

Example 2 (Instruments)**.** *In the context of the previous example, a common identification approach is based on using instrumental variables. Suppose that* Y_{i3} *is an instrument that correlates with with the supply shock but is believed to be uncorrelated with demand, an assumption that we would like to test. After re-defining the errors and parameters we can write the model as*

$$\begin{aligned} Y_{i1} &= \alpha_1 Y_{i2} + \sigma_1 \epsilon_{i1} \\ Y_{i1} &= \sigma_4 Y_{i2} + \sigma_5 Y_{i,3} + \sigma_2 \epsilon_{i2} \\ Y_{i3} &= \alpha_2 \epsilon_{i,1} + \sigma_3 \epsilon_{i3} \end{aligned}$$

where $\alpha_2 = 0$ *implies that the instrument is exogenous and* $\sigma_5 \neq 0$ *implies that the instrument is relevant. We have* $Y_i = (Y_{i1}, Y_{i2}, Y_{i3})'$, $\epsilon_i = (\epsilon_{i1}, \epsilon_{i2}, \epsilon_{i3})'$ *and*

$$A(\alpha, \sigma) = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ \alpha_2 & 0 & \sigma_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\alpha_1 & 0 \\ 1 & -\sigma_4 & -\sigma_5 \\ 0 & 0 & 1 \end{bmatrix} .$$

With this parametrization we have several options. First, assuming that the instruments are exogenous we set $\alpha_2 = 0$, *and use the non-Gaussian errors to provide additional identifying information for* α_1. *This could be of use when instruments are weak. Second, we can relax the instrument exogeneity assumption and jointly test* $\alpha = (\alpha_1, \alpha_2)$. *This allows to simultaneously asses the slope of the demand curve and the exogeneity of the instrument. If the instruments are irrelevant and the errors are Gaussian we will not be able to reject any value.*

Example 3 (Rotation matrix)**.** *As in Gouriéroux, Monfort and Renne (2017) we can set* $A(\alpha, \sigma)^{-1} = \Sigma^{1/2}(\sigma) R(\alpha)'$, *where* $\Sigma^{1/2}(\sigma)$ *is lower triangular with parameters* σ *and* $R(\alpha)$ *is a rotation matrix. In this setting we have* $\sigma = \text{vech}(\Sigma^{1/2})$ *and* α *parametrizes* R *using the trigonometric transformation, the Cayley transformation or the exponential transformation of a skew-symmetric matrix (e.g. Gouriéroux, Monfort and Renne, 2017; Magnus, Pijls and Sentana, 2021).*

These examples highlight different options for parametrizing $A(\alpha, \sigma)$. In examples 1 and

2 the parameter α_1 has a direct economic interpretation after an economically interesting permutation has been selected using either sign restrictions or the instrumental variable. In example 2 the parameter α_2 has an econometrically interesting interpretation: if it is non-zero, the instrument is not exogenous. In example, 3 the parameters α do not have a direct structural interpretation, but this specification corresponds to a common choice in the ICA literature (e.g. Hyvärinen, Karhunen and Oja, 2001; Gouriéroux, Monfort and Renne, 2017).

3.2 Effective score test for LSEMs

Next, we provide a step by step implementation guide for the semi-parametric score test that aims to test $H_0 = \alpha = \alpha_0$. We postpone the theoretical justification of the test to the next section.

Effective score and information matrix As intuitively explained in the simple example of Section 2, the proposed score test for the null hypothesis $H_0 : \alpha = \alpha_0$ is of the Neyman-Rao type, which relies on the effective scores for the parameters of interest α. Loosely speaking these scores are defined as the projection of the score function for α on the orthogonal complement of the space spanned by the score functions for the nuisance parameters (β, η) (e.g. Choi, Hall and Schick, 1996; Bickel et al., 1998; Newey, 1990; van der Vaart, 2002).

In the case of interest here, where the nuisance parameter contains both finite (β) and infinite-dimensional (η) components, the effective score function can be calculated in two steps: (1) compute the projection of the score for $\gamma = (\alpha, \beta)$ on the orthocomplement of the space spanned by the score functions for η, and (2) partition the resulting object into the components corresponding to α and β and project the former onto the orthocomplement of the latter.

For step (1) we follow Amari and Cardoso (1997) and Chen and Bickel (2006) who derive this projection for a special case of the LSEM (3) where B is known to be 0, i.e. the ICA model. The log likelihood contribution for observation W_i from model (3) is given by

$$\ell_\theta(W_i) = \log|A| + \sum_{k=1}^{K} \log \eta_k(A_{k\bullet}V_i) + \log \eta_0(\tilde{X}_i) \ ,$$

where $V_i = Y_i - BX_i$.[6] The scores (i.e. partial derivatives of ℓ_θ) with respect to the components of α, σ and b are denoted by $\dot{\ell}_{\theta,\alpha_l} = \nabla_{\alpha_l}\ell_\theta$, $\dot{\ell}_{\theta,\sigma_l} = \nabla_{\sigma_l}\ell_\theta$ and $\dot{\ell}_{\theta,b_l} = \nabla_{b_l}\ell_\theta$. The effective scores are obtained by projecting $\dot{\ell}_{\theta,\alpha_l}, \dot{\ell}_{\theta,\sigma_l}$ and $\dot{\ell}_{\theta,b_l}$ on the orthocomplement of the

[6] Throughout the main text the dependence of e.g. V_i, A, $D_{x,l}$ and $\zeta^x_{l,k,j}$, with $x \in \{\alpha, \sigma\}$, on (parts of) γ is left implicit.

space spanned by the score functions for η:[7]

$$\mathcal{T} = \left\{ w \mapsto h_0(\tilde{x}) + \sum_{k=1}^{K} h_k(A_{k\bullet}(y - Bx)) : h = (h_0, h_1, \ldots, h_K) \in H = \prod_{k=0}^{K} H_k \right\} \tag{6}$$

where $x = (1, \tilde{x}')'$, $w = (y', x')'$. H_0 is the space of bounded functions $h_0 : \mathbb{R}^{d-1} \to \mathbb{R}$ which satisfy $\mathbb{E}h_0(\tilde{X}_i) = 0$. For $k = 1, \ldots, K$, H_k is the space of functions $h_k : \mathbb{R} \to \mathbb{R}$ which are bounded and continuously differentiable with bounded derivative and satisfy $\mathbb{E}[h_k(\epsilon_{i,k})] = \mathbb{E}[\epsilon_{i,k} h_k(\epsilon_{i,k})] = \mathbb{E}[\kappa(\epsilon_{i,k}) h_k(\epsilon_{i,k})] = 0$, with $\kappa(z) = 1 - z^2$. The set $\mathcal{T}$ is the collection of scores corresponding to $\eta = (\eta_0, \eta_1, \ldots, \eta_K)$: the densities of $\tilde{X}_i$ and $\epsilon_{i1}, \ldots, \epsilon_{iK}$, see Lemma 1 in the appendix for a formal statement.

Intuitively, each $h_k \in H_k$ is restricted such that $\eta_k(1 + th_k)$ is a density function and satisfies the conditions imposed by the model (for all small enough t). For instance, for $k = 1, \ldots, K$, the restrictions on h_k ensure that $\epsilon_{ik} = A_{k\bullet} V_i$ remains mean zero and with variance one under the density $\eta_k(1+th_k)$. The elements of the set $\mathcal{T}$ are obtained by taking the derivative of the log likelihood evaluated at $\theta_t = (\gamma, \eta_0(1 + th_0), \ldots, \eta_K(1 + th_K))$ with respect to t and evaluating this at $t = 0$, for a given $h = (h_0, \ldots, h_K) \in H$; see van der Vaart (1998, Section 25.3) for a general discussion.

The effective scores are then defined as $\tilde{\ell}_{\theta,\alpha_l} = \dot{\ell}_{\theta,\alpha_l} - \Pi\dot{\ell}_{\theta,\alpha_l}$, $\tilde{\ell}_{\theta,\sigma_l} = \dot{\ell}_{\theta,\sigma_l} - \Pi\dot{\ell}_{\theta,\sigma_l}$ and $\tilde{\ell}_{\theta,b_l} = \dot{\ell}_{\theta,b_l} - \Pi\dot{\ell}_{\theta,b_l}$, where Π denotes the projection on $\operatorname{cl}\mathcal{T}$, the closure of $\mathcal{T}$. We compute these projections analytically to obtain

$$\begin{aligned}
\tilde{\ell}_{\theta,\alpha_l}(W_i) &= \sum_{k=1}^{K} \sum_{j=1, j\neq k}^{K} \zeta^{\alpha}_{l,k,j} \phi_k(A_{k\bullet}V_i) A_{j\bullet}V_i + \sum_{k=1}^{K} \zeta^{\alpha}_{l,k,k} \left[\tau_{k,1} A_{k\bullet}V_i + \tau_{k,2}\kappa(A_{k\bullet}V_i)\right] \\
\tilde{\ell}_{\theta,\sigma_l}(W_i) &= \sum_{k=1}^{K} \sum_{j=1, j\neq k}^{K} \zeta^{\sigma}_{l,k,j} \phi_k(A_{k\bullet}V_i) A_{j\bullet}V_i + \sum_{k=1}^{K} \zeta^{\sigma}_{l,k,k} \left[\tau_{k,1} A_{k\bullet}V_i + \tau_{k,2}\kappa(A_{k\bullet}V_i)\right] \\
\tilde{\ell}_{\theta,b_l}(W_i) &= \sum_{k=1}^{K} [-A_{k\bullet} D_{b,l}] \left[(X_i - \mathbb{E}X_i)\phi_k(A_{k\bullet}V_i) - \mathbb{E}X_i \left(\varsigma_{k,1} A_{k\bullet}V_i + \varsigma_{k,2}\kappa(A_{k\bullet}V_i)\right)\right]
\end{aligned}$$

where $\zeta^{\alpha}_{l,k,j}$ and $\zeta^{\sigma}_{l,k,j}$ are defined in Assumption 1, $D_{b,l} = \partial B / \partial b_l$ and $\phi_k(x) = \partial \log \eta_k(x) / \partial x$. Further,

$$\tau_k = M_k^{-1} \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \quad \varsigma_k = M_k^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{where } M_k = \begin{pmatrix} 1 & \mathbb{E}_\theta(A_{k\bullet}V_i)^3 \\ \mathbb{E}_\theta(A_{k\bullet}V_i)^3 & \mathbb{E}_\theta(A_{k\bullet}V_i)^4 - 1 \end{pmatrix}.$$

The derivations that lead to these expressions are given the appendix where Lemma 3 pro-

[7] Each score function lies in $L_2(P_\theta)$, which is the Hilbert space under consideration here.

vides the formal statement. The expressions show that the effective scores depend on the log density scores ϕ_k, i.e. the non-parametric part stemming from η_k, and the third and fourth moments of the errors $A_{k\bullet}V_i$ via the vectors τ_k and ς_k, for $k = 1, \ldots, K$.

For step (2) we will project the effective scores for α on the space spanned by the effective scores for $\beta = (b, \sigma)$. Since the latter space is finite dimensional this projection takes a standard form. First, we collect and partition the effective scores as follows

$$\tilde{\ell}_\theta(W_i) = \begin{bmatrix} \tilde{\ell}_{\theta,\alpha}(W_i) \\ \tilde{\ell}_{\theta,\beta}(W_i) \end{bmatrix} \qquad \text{and} \qquad \tilde{\ell}_{\theta,\beta}(W_i) = \begin{bmatrix} \tilde{\ell}_{\theta,\sigma}(W_i) \\ \tilde{\ell}_{\theta,b}(W_i) \end{bmatrix} ,$$

where $\tilde{\ell}_{\theta,\alpha} = (\tilde{\ell}_{\theta,\alpha_1}, \ldots, \tilde{\ell}_{\theta,\alpha_{L_\alpha}})'$, $\tilde{\ell}_{\theta,\sigma} = (\tilde{\ell}_{\theta,\sigma_1}, \ldots, \tilde{\ell}_{\theta,\sigma_{L_\sigma}})'$ and $\tilde{\ell}_{\theta,b} = (\tilde{\ell}_{\theta,b_1}, \ldots, \tilde{\ell}_{\theta,b_{L_b}})'$ are the $L_\alpha \times 1$, $L_\sigma \times 1$ and $L_b \times 1$ vectors that collect the effective score functions. With this notation we define the effective information matrix by

$$\tilde{I}_\theta = \mathbb{E}\tilde{\ell}_\theta(W_i)\tilde{\ell}'_\theta(W_i) \qquad \text{with partitioning} \quad \tilde{I}_\theta = \begin{pmatrix} \tilde{I}_{\theta,\alpha\alpha} & \tilde{I}_{\theta,\alpha\beta} \\ \tilde{I}_{\theta,\beta\alpha} & \tilde{I}_{\theta,\beta\beta} \end{pmatrix} .$$

The effective score function for α with respect to β and η can now be computed by the second projection (e.g. Bickel et al., 1998, p. 74)

$$\breve{\kappa}_\theta(W_i) = \tilde{\ell}_{\theta,\alpha}(W_i) - \tilde{I}_{\theta,\alpha\beta}\tilde{I}^{-1}_{\theta,\beta\beta}\tilde{\ell}_{\theta,\beta}(W_i) \ . \tag{7}$$

The corresponding effective information matrix is given by

$$\tilde{\mathcal{I}}_\theta = \tilde{I}_{\theta,\alpha\alpha} - \tilde{I}_{\theta,\alpha\beta}\tilde{I}^{-1}_{\theta,\beta\beta}\tilde{I}_{\theta,\beta\alpha} \ . \tag{8}$$

We note that the effective score function $\breve{\kappa}_\theta(W_i)$ and the effective information matrix $\tilde{\mathcal{I}}_\theta$ can be evaluated at any parameters $\theta = (\alpha, \beta, \eta)$.

Effective score and information matrix estimation The effective scores and information depend on unknown nuisance parameters, such as the log density scores ϕ_k and the moment vectors τ_k and ζ_k. To implement the score test we replace these parameters by appropriate estimates. As we show in the appendix, consistent estimators for $\tilde{\ell}_\theta(W_i)$ are

$$\hat{\ell}_\gamma(W_i) = \begin{bmatrix} \hat{\ell}_{\gamma,\alpha}(W_i) \\ \hat{\ell}_{\gamma,\beta}(W_i) \end{bmatrix} \qquad \text{and} \qquad \hat{\ell}_{\gamma,\beta}(W_i) = \begin{bmatrix} \hat{\ell}_{\gamma,\sigma}(W_i) \\ \hat{\ell}_{\theta,b}(W_i) \end{bmatrix} ,$$

where the components are given by

$$\begin{aligned}
\hat{\ell}_{\gamma,\alpha_l}(W_i) &= \sum_{j,k=1,j\neq k}^{K} \zeta^{\alpha}_{l,k,j}\hat{\phi}_k(A_{k\bullet}V_i)A_{j\bullet}V_i + \sum_{k=1}^{K}\zeta^{\alpha}_{l,k,k}\left[\hat{\tau}_{k,1}A_{k\bullet}V_i + \hat{\tau}_{k,2}\kappa(A_{k\bullet}V_i)\right]\\
\hat{\ell}_{\gamma,\sigma_l}(W_i) &= \sum_{j,k=1,j\neq k}^{K} \zeta^{\sigma}_{l,k,j}\hat{\phi}_k(A_{k\bullet}V_i)A_{j\bullet}V_i + \sum_{k=1}^{K}\zeta^{\sigma}_{l,k,k}\left[\hat{\tau}_{k,1}A_{k\bullet}V_i + \hat{\tau}_{k,2}\kappa(A_{k\bullet}V_i)\right]\ , \qquad (9)\\
\hat{\ell}_{\gamma,b_l}(W_i) &= \sum_{k=1}^{K}[-A_{k\bullet}D_{b,l}][(X_i-\bar{X})\hat{\phi}_k(A_{k\bullet}V_i) - \bar{X}(\hat{\varsigma}_{k,1}A_{k\bullet}V_i + \hat{\varsigma}_{k,2}\kappa(A_{k\bullet}V_i))]
\end{aligned}$$

with $\bar{X} = n^{-1}\sum_{i=1}^{n}X_i$. The coefficients $\hat{\tau}_k = (\hat{\tau}_{k,1},\hat{\tau}_{k,2})'$ and $\hat{\varsigma}_k = (\hat{\varsigma}_{k,1},\hat{\varsigma}_{k,2})'$ are given, for $k = 1,\ldots,K$, by

$$\hat{\tau}_k = \hat{M}_k^{-1}\begin{pmatrix}0\\-2\end{pmatrix}\ ,\ \hat{\varsigma}_k = \hat{M}_k^{-1}\begin{pmatrix}1\\0\end{pmatrix}\ ,\ \hat{M}_k = \begin{pmatrix}1 & \frac{1}{n}\sum_{i=1}^{n}(A_{k\bullet}V_i)^3\\ \frac{1}{n}\sum_{i=1}^{n}(A_{k\bullet}V_i)^3 & \frac{1}{n}\sum_{i=1}^{n}(A_{k\bullet}V_i)^4 - 1\end{pmatrix}\ . \qquad (10)$$

The estimates for the effective scores can be evaluated at any $\gamma = (\alpha,\beta)$, but do not depend on ϕ_k, τ_k, ς_k or $\mathbb{E}X_i$ as these components have been replaced by estimators $\hat{\phi}_k$, $\hat{\tau}_k$, $\hat{\varsigma}_k$ and $\bar{X}$. These estimators may depend on γ and the index n, though this is left implicit in the notation.

Density score estimation The log density score estimates $\hat{\phi}_k(\cdot)$ needed for computing (9) can be obtained in different ways and our preferred approach is based on using B-splines as in Jin (1992) and Chen and Bickel (2006). We can define these estimates by

$$\hat{\phi}_k(z) = \hat{\psi}_k' b_k(z)\ ,\ \text{with}\quad \hat{\psi}_k = -\left[\sum_{i=1}^{n} b_k(A_{k\bullet}V_i)b_k(A_{k\bullet}V_i)'\right]^{-1}\sum_{i=1}^{n} c_k(A_{k\bullet}V_i)\ , \qquad (11)$$

where z is the argument of the function, e.g. $z = A_{k\bullet}V_i$ in (9), $b_k(z) = (b_{k,1}(z),\ldots,b_{k,\mathsf{B}_k}(z))'$ is a collection of B_k cubic B-splines and $c_k(z) = (c_{k,1}(z),\ldots,c_{k,\mathsf{B}_k}(z))'$ are their derivatives: $c_{k,i}(z) = \frac{\mathrm{d}b_{k,i}(z)}{\mathrm{d}z}$ for each $i = 1,\ldots,\mathsf{B}_k$, see de Boor (2001) for more details on B-splines.[8] In practice we rely on equally spaced knots with upper and lower end points taken to be the 95th and 5th percentile of the samples $\{\epsilon_i\}_{i=1}^{n}$ adjusted by $\log(\log(n))$. We use $\mathsf{B}_k = 6$ splines in our main simulations below and investigate the sensitivity of this choice.

Given the estimates of the effective scores we estimate the effective information matrix,

[8] Further details as required for the construction in this paper are given in Section S5 in the supplementary material. For the asymptotic theory, B_k will be required to (slowly) diverge with n. In the main text we omit the dependence of B_k and b_k on n in the notation.

which is the variance matrix of the effective score function, as

$$\hat{I}_\gamma = \frac{1}{n}\sum_{i=1}^{n} \hat{\ell}_\gamma(W_i)\hat{\ell}_\gamma(W_i)' \qquad \text{with partitioning} \quad \hat{I}_\gamma = \begin{bmatrix} \hat{I}_{\gamma,\alpha\alpha} & \hat{I}_{\gamma,\alpha\beta} \\ \hat{I}_{\gamma,\beta\alpha} & \hat{I}_{\gamma,\beta\beta} \end{bmatrix} . \tag{12}$$

With these estimates we can compute the estimates for the effective score of α with respect to β and η, i.e. $\tilde{\kappa}_\theta(W_i)$ as defined in (7), and the corresponding information matrix (8).

$$\hat{\kappa}_\gamma(W_i) = \hat{\ell}_{\gamma,\alpha}(W_i) - \hat{I}_{\gamma,\alpha\beta}\hat{I}^{-1}_{\gamma,\beta\beta}\hat{\ell}_{\gamma,\beta}(W_i) \qquad \text{and} \qquad \hat{\mathcal{I}}_\gamma = \hat{I}_{\gamma,\alpha\alpha} - \hat{I}_{\gamma,\alpha\beta}\hat{I}^{-1}_{\gamma,\beta\beta}\hat{I}_{\gamma,\beta\alpha} . \tag{13}$$

Importantly, $\tilde{\mathcal{I}}_\theta$ may not be positive definite in our setting. For instance, when the densities η_k correspond to the Gaussian density, $\tilde{\mathcal{I}}_\theta$ is singular, see the discussion preceding Lemma S15 in the supplementary material.

Semi-parametric score statistic With $\hat{\kappa}_\gamma$ and $\hat{\mathcal{I}}_\gamma$ we can define the semi-parametric score test statistic for the LSEM model as function of $\gamma = (\alpha, \beta)$ and the observations W_i by

$$\hat{S}_\gamma = \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{\kappa}_\gamma(W_i)\right)' \hat{\mathcal{I}}^{t,\dagger}_\gamma \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\hat{\kappa}_\gamma(W_i)\right) , \tag{14}$$

where $\hat{\mathcal{I}}^{t,\dagger}_\gamma$ denotes the generalized inverse of the eigenvalue truncated effective information matrix $\hat{\mathcal{I}}_\gamma$ (cf. Lütkepohl and Burda, 1997). Formally,

$$\hat{\mathcal{I}}^{t}_\gamma = \hat{U}\hat{\Lambda}(\nu_n^{1/2})\hat{U}' , \tag{15}$$

where $\hat{\Lambda}(\nu_n^{1/2})$ is a diagonal matrix with the $\nu_n^{1/2}$-truncated eigenvalues of $\hat{\mathcal{I}}_\gamma$ on the main diagonal and $\hat{U}$ is the matrix of corresponding orthonormal eigenvectors. To be specific, let $\{\hat{\lambda}_i\}_{i=1}^{L}$ denote the non-increasing eigenvalues of $\hat{\mathcal{I}}_\gamma$, then the (i,i)th element of $\hat{\Lambda}(\nu_n^{1/2})$ is given by $\hat{\lambda}_i \mathbf{1}(\hat{\lambda}_i \geq \nu_n^{1/2})$. We discuss the choice for the truncation parameter in more detail below.

Equations (9)-(15) define the semi-parametric score statistic for the LSEM model (3) for a given parameter vector $\gamma = (\alpha, \beta)$. To test the null hypothesis (5) we will evaluate this test statistic at $\alpha = \alpha_0$, i.e. fixing the possibly unidentified parameters under the null, and at $\hat{\beta}$, which can be any $\sqrt{n}$ consistent estimate for β. Let $\hat{\gamma} = (\alpha_0, \hat{\beta})$. In our simulations, we use ordinary least squares estimates for σ and $b = \text{vec}(B)$, or one-step efficient estimates following van der Vaart (2002, Section 7.2). In our theoretical section below we show that under suitable assumptions the score statistic will converge to a χ^2 limit. Specifically, we prove that under H_0 for any $a \in (0,1)$ we have

$$\lim_{n\to\infty} P(\hat{S}_{\hat{\gamma}} > c_n) \leq a \ , \tag{16}$$

where c_n is the $1-a$ quantile of the $\chi^2_{r_n}$ distribution with $r_n = \text{rank}(\hat{\mathcal{I}}^t_{\hat{\gamma}})$. Importantly, as we show in section 4 this result does not rely on any assumptions regarding the shape of the densities η, i.e. we do not need to assume that η is non-Gaussian. Only conventional moment assumptions and some regularity conditions on the densities are required. The following algorithm summarizes the complete implementation.

Algorithm: Effective score test for LSEM

```
1 Obtain $\sqrt{n}$-consistent estimates $\hat{\beta} = (\hat{\sigma}, \hat{b})$, residuals $\hat{V}_i = Y_i - \hat{B}X_i$ and evaluate all quantities in steps 2-5 at $\hat{\gamma} = (\alpha_0, \hat{\beta})$;
2 For $k = 1, \ldots, K$, compute $\hat{\phi}_k(\hat{A}_{k\bullet}\hat{V}_i)$ from (11) with $\hat{A} = A(\alpha_0, \hat{\sigma})$;
3 Compute the effective scores $\hat{\ell}_{\hat{\gamma}}(W_i)$ from (9) and the information matrix $\hat{I}_{\hat{\gamma}}$ from (12);
4 Compute $\hat{\kappa}_{\hat{\gamma}}(W_i)$ and $\hat{\mathcal{I}}_{\hat{\gamma}}$ from (13) and $\hat{\mathcal{I}}^t_{\hat{\gamma}}$ from (15) using truncation parameter $\nu_n^{1/2}$;
5 Compute the score statistic $\hat{S}_{\hat{\gamma}}$ from (14) and reject $H_0 : \alpha = \alpha_0$ if $\hat{S}_{\hat{\gamma}} > c_n$, where $c_n$ is the $1-a$ quantile of the $\chi^2_{r_n}$ distribution with $r_n = \text{rank}(\hat{\mathcal{I}}^t_{\hat{\gamma}})$.
```

The truncation parameter $\nu_n^{1/2}$ in step 4 is a tuning parameter for which the theoretical requirements are formalized in Assumption 3 below. In practice, we recommend a small tuning parameter (e.g. less than $\nu_n^{1/2} = 10^{-5}$) as our simulations suggest that the null rejection probability is well controlled for any such choice.[9] In practice the simplest implementation is to use a pseudo inverse function directly which implicitly truncates at machine precision. Nevertheless we recommend that researchers applying the proposed approach explore the performance of different choices of $\nu_n^{1/2}$ in simulation experiments designed to replicate the application at hand.

The algorithm highlights that the computational cost for evaluating the semi-parametric score statistic $\hat{S}_{\hat{\gamma}}$ is modest; effectively one only needs to compute K B-spline regressions

[9] See Section S7.1 in the supplementary material for simulation results with different truncation values.

to obtain the log density scores. Importantly, this implies that the algorithm can often implemented without relying on numerical optimization routines.[10] Confidence sets for α can be constructed by inverting the score statistic over a range of values for α_0.

For some parametrizations of $A(\alpha, \sigma)$, the parameter of economic interest could be a function of both α and σ, or more generally, a function of α and $\beta = (b, \sigma)$. In these settings, the algorithm can be used in combination with the Bonferroni approach discussed in Granziera, Moon and Schorfheide (2018) to construct confidence intervals for such functions. Intuitively, this approach amounts to constructing a confidence set for $f(\alpha_0, \hat{\beta})$ with confidence level q_2 for each fixed α_0 for which the score test does not reject at level q_1, with $q_1 + q_2 = a$. Then, taking the union over the constructed sets for $f(\alpha_0, \hat{\beta})$ yields a $1 - a$ confidence set for $f(\alpha, \beta)$.

4 Asymptotic theory

In this section we present our main theoretical results. We start by carefully spelling out the regularity conditions that are required. After this we discuss the properties of the test. We show that (i) under weak identification asymptotics, the null rejection probability of the test does not exceed its nominal level asymptotically and (ii) under strong identification it attains well known power bounds for various classes of tests.

4.1 Assumptions

We assume that we observe a random sample $\{W_i\}_{i=1}^n = \{(Y_i', X_i')'\}_{i=1}^n$ from model (3) where the underlying components satisfy the following.

Assumption 2. *For $\epsilon_i = (\epsilon_{i1}, \ldots, \epsilon_{iK})'$ in model (3), each component ϵ_{ik} has a continuously differentiable root density (with respect to Lebesgue measure on $\mathbb{R}$). We write the density as η_k with log density score $\phi_k(x) = \partial \log \eta_k(x)/\partial x$. We assume that for all $k = 1, \ldots, K$ and some $\delta > 0$*

1. *$\mathbb{E}\epsilon_{ik} = 0$, $\mathbb{E}\epsilon_{ik}^2 = 1$, $\mathbb{E}\epsilon_{ik}^{4+\delta} < \infty$, $\mathbb{E}(\epsilon_{ik}^4) - 1 > \mathbb{E}(\epsilon_{ik}^3)^2$, and $\mathbb{E}\phi_k^{4+\delta}(\epsilon_{ik}) < \infty$;*
2. *$\mathbb{E}\phi_k(\epsilon_{ik}) = 0$, $\mathbb{E}\phi_k(\epsilon_{ik})\epsilon_{ik} = -1$, $\mathbb{E}\phi_k(\epsilon_{ik})\epsilon_{ik}^2 = 0$ and $\mathbb{E}\phi_k(\epsilon_{ik})\epsilon_{ik}^3 = -3$;*
3. *ϵ_{ik} is independent of ϵ_{il} for all $k \neq l$;*

[10] Numerical optimisation may be necessary to compute $\hat{\beta}$, depending on the chosen parametrisation, but is not necessary beyond this.

4. *$\eta_0 \in \mathscr{X}$ is a density function (with respect to Lebesgue measure on $\mathbb{R}^{d-1}$) such that if $\tilde{X}_i \sim \eta_0$, then $\mathbb{E}\tilde{X}_i\tilde{X}_i'$ is positive definite and $\mathbb{E}[|\tilde{X}_{i,l}|^{4+\delta}] < \infty$ for all $l = 1, \ldots, d-1$;*

5. *ϵ_i and $\tilde{X}_i$ are independent.*

The first part normalizes the errors to have mean zero, variance one and finite four+δ moments,[11] hence ruling out heavy tailed errors.[12] Additionally, we require the log density scores $\phi_k(x) = \partial \log \eta_k(x)/\partial x$ evaluated at the errors to have finite four+δ moments. The second part simplifies the construction of the effective score functions. Whilst this may at first glance appear a strong condition, Lemma S16 in the supplementary material shows that if the first part holds, then a simple sufficient condition is that the tails of the densities η_k converge to zero at a polynomial rate.[13] The third part imposes that the components of ϵ_i are independent. Part four imposes some structure on $\tilde{X}_i$ that allows us to identify B; notably positive definite second moments and four+δ finite moments are required. Part five requires the explanatory variables and errors to be independent. This can be relaxed by requiring the moment assumptions in 2 to hold conditional on $\tilde{X}_i$. In this setup, our general theory as outlined in this section would continue to be valid though the resulting effective score function would take a different form.

Most important is what is *not* in Assumption 2: there is no condition that imposes that a certain number of components of ϵ_i have a (sufficiently) non-Gaussian distribution.

The third assumption that we impose is only required for the estimation of the log density scores $\phi_k(x) = \partial \log \eta_k(x)/\partial x$ using B-spline regressions and can be appropriately replaced when a different density score estimator is used.[14] For notation purposes, let $\Xi^L_{k,n}$ and $\Xi^U_{k,n}$ denote the lower and upper endpoints of the cubic B-splines for $\phi_k(x)$ for $k = 1, \ldots, K$. In practice, we select these points as the lower 5th and upper 95th percentiles of the samples $\{A_{k\bullet}V_i\}_{i=1}^n$ adjusted by $\log\log n$, see the implementation section 3.

Assumption 3. *Let ν_n be such that $\nu_{n,p}^2 = o(\nu_n)$ with $p := \min\{1+\delta/4, 2\}$ and $\nu_{n,p} = n^{(1-p)/p}$ if $p \in (1,2)$ or $\nu_{n,p} = n^{-1/2}\log(n)^{1/2+\rho}$, for some $\rho > 0$, if $p = 2$. Let $\phi_{k,n} := \phi_k \mathbf{1}_{[\Xi^L_{k,n}, \Xi^U_{k,n}]}$ and $\Delta_{k,n} := \Xi^U_{k,n} - \Xi^L_{k,n}$ and suppose that for all $k = 1, \ldots, K$, $[\Xi^L_{k,n}, \Xi^U_{k,n}] \uparrow \tilde{\Xi} \supset \operatorname{supp}(\eta_k)$ and $\delta_{k,n} \downarrow 0$*

[11] $\mathbb{E}(\epsilon_{ik}^4) - 1 \geq \mathbb{E}(\epsilon_{ik}^3)^2$ always holds; this is known as Pearson's inequality. See e.g. result 1 in Sen (2012). Assuming that $\mathbb{E}(\epsilon_{ik}^4) - 1 > \mathbb{E}(\epsilon_{ik}^3)^2$ rules out (only) cases where $1, \epsilon_{ik}$ and ϵ_{ik}^2 are linearly dependent when considered as elements of L_2. See e.g. Theorem 7.2.10 in Horn and Johnson (2013).

[12] Heavy tailed errors in ICA and SVAR models have recently been considered in Davis and Ng (2022) and Davis and Fernandes (2022), but an inferential theory remains to be developed.

[13] See Example S1 in the supplementary material for an explicit example of a density which satisfies the first part of the assumption but not the second.

[14] See Assumption 4 for conditions on any alternative density score estimator under which our Theorem 1 continues to hold.

(i) $P(\epsilon_{ik} \notin [\Xi^L_{k,n}, \Xi^U_{k,n}]) = o(\nu_n^2)$*;*

(ii) *For some* $\iota > 0$, $n^{-1}\Delta_{k,n}^{2+2\iota}\delta_{k,n}^{-(8+2\iota)} = o(\nu_n)$*;*

(iii) η_k *is bounded* $(\|\eta_k\|_\infty < \infty)$ *and differentiable, with a bounded derivative:* $\|\eta_k'\|_\infty < \infty$*;*

(iv) *For each* n, $\phi_{k,n}$ *is three-times continuously differentiable on* $[\Xi^L_{k,n}, \Xi^U_{k,n}]$ *and* $\|\phi^{(3)}_{k,n}\|^2_\infty \delta^6_{k,n} = o(\nu_n)$*;*[15]

(v) *There are* $c > 0$ *and* $N \in \mathbb{N}$ *such that for* $n \geq N$ *we have* $\inf_{t\in[\Xi^L_{k,n},\Xi^U_{k,n}]} |\eta_k(t)| \geq c\delta_{k,n}$.

First, the assumption provides conditions on the truncation rate $\nu_n^{1/2}$ that is needed for the truncation of the eigenvalues in (15). This rate is split into two parts. The "slow" rate $n^{(1-p)/p}$ (for $p \in (1, 2)$) is always sufficient given assumption 2, but if ϵ_{ik} has finite eighth moments the faster rate applies.

Part (i) imposes that the tails of ϵ_{ik} decay to zero sufficiently fast.[16] Part (ii) ensures that the number of knots does not grow to fast relative to the sample size (and the truncation rate). Part (iii) requires the density and its derivative to be bounded. Part (iv) requires the existence of the third derivatives of ϕ_k and that the rate of increase of the third derivative is not too great. Part (v) ensures that the density is bounded away from zero on $[\Xi^L_{k,n}, \Xi^U_{k,n}]$. Overall, these assumptions are similar to those adopted in Chen and Bickel (2006), with two key differences.[17] Firstly, Chen and Bickel (2006) require the conditions to hold for the functions $v \mapsto \phi_k(A_{k\bullet}v)$ (rather than ϕ_k), uniformly over shrinking balls (at rate $n^{-1/2}$) around A. In our setting we are only interested in testing as consistent estimation is ruled out by the possible lack of identification, hence we only require the conditions to hold for the functions ϕ_k. Secondly, unlike Chen and Bickel (2006), we require convergence at a rate ν_n which satisfies certain decay conditions. This is due to the fact that we may have a singular effective information matrix and in order to obtain a consistent estimate of the Moore – Penrose inverse of this matrix, we require knowledge of the rate of convergence of our estimator.

4.2 Main results

In this section we formally state our main results for the semi-parametric score test $\hat{S}_{\hat{\gamma}}$. First, instead of evaluating the score test at the $\sqrt{n}$-consistent estimates $\hat{\gamma} = (\alpha_0, \hat{\beta})$ we will

[15] The differentiability and continuity requirements at the end-points are one-sided.

[16] The required speed of decay is linked to the truncation rate.

[17] Cf. their conditions C3, C5 – C7, p. 2834.

evaluate the score test at its discretized version $\bar{\gamma} = (\alpha_0, \bar{\beta}_n)$. Formally, let $\mathsf{G}_n = n^{-1/2} C \mathbb{Z}^{L_\beta}$ for some $C > 0$ and define $\bar{\beta}_n$ as a new version of $\hat{\beta}$ that replaces its value with the closest point in G_n. Note that this changes each coordinate of $\hat{\beta}$ by a quantity which is at most $O(n^{-1/2})$, hence the $\sqrt{n}$-consistency is retained by discretization. Since the constant C can be chosen arbitrarily small this change has no practical relevance for the implementation of the test.

The advantage of relying on discretized estimates is that it simplifies the proof of the main result. Specifically, it removes the need to show uniform convergence between the effective scores evaluated at $\hat{\beta}$ and β. The discretization trick is due to Le Cam (1960) and is widely used in statistics, see the detailed discussion in Le Cam and Yang (2000, Section 6.3), or van der Vaart (1998, page 72).[18]

The following theorem provides the main result.[19]

Theorem 1. *Suppose that Assumptions 1, 2 and 3 hold and that (α_0, β) is an interior point of $\mathcal{A} \times \mathcal{B}$. Let $r_n = \operatorname{rank}(\hat{\mathcal{I}}^t_{\bar{\gamma}})$ and denote by c_n the $1 - a$ quantile of the $\chi^2_{r_n}$ distribution, for any $a \in (0, 1)$. Then for any sequence*

$$\theta_n = (\alpha_0, \beta + d_n/\sqrt{n}, \eta(1 + h_n/\sqrt{n}) \ , \quad d_n \in D^\star \ , \ h_n \in H^\star \ ,$$

with $D^\star$ a bounded subset of $\mathbb{R}^{L_\beta}$ and $H^\star$ a compact subset of H, we have

$$\limsup_{n\to\infty} P^n_{\theta_n}(\hat{S}_{\bar{\gamma}} > c_n) \leq a,$$

with inequality only if $\operatorname{rank}(\tilde{\mathcal{I}}_{\theta_0}) = 0$ where $\theta_0 = (\alpha_0, \beta, \eta)$. The notation $P^n_{\theta_n}$ indicates the n-fold product of the measure P_{θ_n}, i.e. the distribution of the data $W_1, \ldots, W_n$ under θ_n.

Theorem 1 shows that the test is locally robust in that its null rejection probability is no greater than the nominal a under any local sequence θ_n (consistent with the null). Under such sequences, the densities of the structural shocks (i.e. ϵ_{ik}) may converge to the Gaussian density at a $\sqrt{n}$ rate, i.e. these are local-to-Gaussian sequences. Studying the behavior of tests under these local-to-Gaussian sequences is the natural counterpart (in the model we study) to studying the performance of tests under so-called "weak identification asymptotics", as has been considered in many settings (e.g. Staiger and Stock, 1997; Stock and Wright, 2000; Moreira, 2003; Kleibergen, 2005; Andrews and Mikusheva, 2015). The key difference in our setting is that the identification failure occurs due to the value of an

[18] It has also been adopted in econometrics, see Cattaneo, Crump and Jansson (2012) for instance.

[19] The set H which apears in the statement of Theorem 1 is defined in Section 3. See equation (6) and the paragraph following it.

infinite dimensional nuisance parameter.

This local robustness follows from the fact that the test statistic $\hat{S}_{\bar{\gamma}}$ is locally regular, i.e. it attains its limiting distribution (under the null) in a locally uniform manner. This property, in turn, follows from the orthogonalization with respect to (all of) the nuisance parameters in the definition of the effective score function.[20] This orthogonalization ensures that the test statistic is insensitive to small deviations in the nuisance parameters and therefore that its limiting distribution does not change when the limit is taken along sequences of local alternatives consistent with the null hypothesis.

The result of Theorem 1 can be also written as

$$\limsup_{n\to\infty} \sup_{\theta\in\Theta_{0,n}} P_\theta^n(\hat{S}_{\bar{\gamma}} > c_n) \leq a \ ,$$

where

$$\Theta_{0,n} = \{(\alpha_0, \beta + d/\sqrt{n}, \eta(1 + h/\sqrt{n}) : d \in D^\star, h \in H^\star\} \ .$$

This formulation allows us to highlight a difference between our *local* uniformity result, which is over local sets $\Theta_{0,n}$, and a more demanding *global* uniformity result in which the supremum would be taken over $\Theta_0 = \{(\alpha_0, \beta, \eta) : \beta \in \mathcal{B}, \eta \in \mathcal{H}\}$. We emphasise that Theorem 1 does not establish such a result.[21]

Efficiency under strong identification Importantly, the local robustness of the score test does not come at the expense of power loss under strong identification. In particular, the test $\varphi_n \coloneqq \mathbf{1}\{\hat{S}_{\bar{\gamma}} > c_n\}$ is semiparametrically efficient when $\tilde{\mathcal{I}}_\theta$ is nonsingular.[22] Here we provide a brief heuristic discussion of this point; proofs that these power bounds are attained by φ_n can be found in Section S6 of the supplementary appendix.

For the parameters $\theta = (\alpha, \beta, \eta)$ we consider local alternatives of the type

$$\theta_n(q, d, h) = \left(\alpha + q/\sqrt{n},\ \beta + d/\sqrt{n},\ \eta(1 + h/\sqrt{n})\right) \ . \tag{17}$$

First suppose that α is scalar and $\tilde{\mathcal{I}}_\theta > 0$. Then the asymptotic power of the proposed test is against the local alternatives in (17) is

$$\lim_{n\to\infty} P^n_{\theta_n(q,d,h)}\varphi_n = 1 - \Phi\left(z_{a/2} - \tilde{\mathcal{I}}_\theta^{1/2} q\right) + 1 - \Phi\left(z_{a/2} + \tilde{\mathcal{I}}_\theta^{1/2} q\right) \ , \tag{18}$$

[20] In conjunction with the ULAN property shown to hold in Lemma 2.

[21] For models where identification failures are determined by a finite dimensional η, global uniformity conditions are derived in Andrews and Cheng (2012, 2013) and Andrews, Cheng and Guggenberger (2020). For the case where η is infinite dimensional much work remains to be done.

[22] Nonsingularity may fail to hold when multiple components of ϵ_i are Gaussian; see Lemma S15.

where Φ the standard normal CDF and $z_{a/2}$ the $1-a/2$ quantile of the $\mathcal{N}(0,1)$. This coincides with the (local asymptotic) power bound for locally asymptotically unbiased two sided tests of $q=0$ against $q\neq 0$ (cf. Theorem 2 in Choi, Hall and Schick (1996)).[23,24]

If instead α is multidimensional and $\tilde{\mathcal{I}}_\theta$ is positive definite, then the asymptotic power of the proposed test is against the local alternatives in (17) is

$$\lim_{n\to\infty} P^n_{\theta_n(q,d,h)}\varphi_n = 1 - \mathrm{P}\left(\chi^2_{L_\alpha}(q'\tilde{\mathcal{I}}_\theta q) \leq c_a\right), \tag{19}$$

where $\chi^2_r(u)$ denotes a random variable with a non-central χ^2 distribution with r degrees of freedom and non-centrality parameter u and c_a is the $1-a$ quantile of the (central) $\chi^2_{L_\alpha}$ distribution. This coincides with the (local asymptotic) power bound for asymptotically rotation invariant tests as developed in Section 5 of Choi, Hall and Schick (1996) (see their Theorem 3).[25,26]

These power bounds make φ_n attractive in scenarios where there is no explicit direction in which one want to maximize power. When such directions are given alternative test statistics, also based on the effective score function, can be considered (e.g. Bickel, Ritov and Stoker, 2006). Maximin optimality results which permit singular $\tilde{\mathcal{I}}_\theta$ matrices can be found in Lee (2023) for related tests in general semi-parametric models.

5 Simulation results

In this section we study the finite sample properties of the semi-parametric score test $\hat{S}_{\hat{\gamma}}$. We study the empirical rejection frequency of the test under different data generating processes

[23] One can alternatively see this by approximating the infinite dimensional model by a sequence of finite-dimenional models for which the corresponding result is well known and then taking limits. Cf. the proof of Theorem 25.44 in van der Vaart (1998).

[24] That the sequence of tests $(\varphi_n)_{n\in\mathbb{N}}$ is itself locally asymptotically unbiased is clear from (18).

[25] That the sequence of tests $(\varphi_n)_{n\in\mathbb{N}}$ is itself asymptotically rotation invariant is clear from (19): the limiting power function is that of the test $\varphi(Z) \coloneqq \mathbf{1}\{Z'Z > c_a\}$ for $Z\sim\mathcal{N}(\tilde{\mathcal{I}}_\theta^{1/2}q, I)$. This test is rotation invariant since for any rotation matrix R and any $z\in\mathbb{R}^{L_\alpha}$ one has $\varphi(R'z) = \mathbf{1}\{z'RR'z > c_a\} = \mathbf{1}\{z'z > c_a\} = \varphi(z)$.

[26] Related, the asymptotic maximin power of φ_n against the alternatives in (17) is

$$\lim_{n\to\infty} \inf_{(q,d,h)\in K^\star_u} P^n_{\theta_n(q,d,h)}\phi_n = 1 - \mathrm{P}\left(\chi^2_{L_\alpha}(u) \leq c_a\right), \tag{20}$$

where $K^\star_u$ is any compact subset of

$$K_u \coloneqq \left\{(q,d,h)\in\mathbb{R}^{L_\alpha}\times\mathbb{R}^{L_\beta}\times H : q'\tilde{\mathcal{I}}_\theta q \geq u\right\}.$$

which also coincides with the (local asymptotic) maximin power bound (cf. the parametric case in Theorem 13.5.5 of Lehmann and Romano (2005)).

and compare its performance to several alternatives that have been proposed in the literature. We first study the simple model of section (2) after which we consider the general linear simultaneous equations model (3). The supplementary material provides additional results.

5.1 Baseline model

We start by drawing independent samples from model (1), which we restate for convenience

$$Y_i = A^{-1}\epsilon_i \ , \qquad i = 1, \ldots, n \ .$$

We take Y_i to be $K \times 1$ and consider $K = 2, 3$ and $K = 5$. The sample size is taken as $n = 200, 500$ or $n = 1000$. We fix ϵ_{i1} to have a standard Gaussian density and consider different densities for ϵ_{ik}, with $k = 2, \ldots, K$. The non-Gaussian densities are either Student's t or mixtures of normals taken from Marron and Wand (1992). Figure 3 provides an overview.

The matrix of interest A is taken as a rotation matrix and parametrized by the Cayley transformation of a skew-symmetric matrix (e.g. Gouriéroux, Monfort and Renne, 2017):

$$A = A(\alpha) = (I - \Omega(\alpha))(I + \Omega(\alpha))^{-1} \ ,$$

where $\Omega(\alpha)$ is a skew-symmetric matrix (i.e. $\Omega(\alpha)' = -\Omega(\alpha)$) parameterized by α which we sample at random from $\alpha \sim N(0, I_{L_\alpha})$.

In this setting there are no additional nuisance parameters which allows us to concentrate on the consequences of weak non-Gaussianity on the semi-parametric score test and some alternative tests that have been proposed in the literature. In the simulation designs below we include additional finite dimensional nuisance parameters (i.e. $\beta = (b, \sigma)$) and investigate whether their inclusion alters the empirical rejection frequency of the test.

For each specification we simulate $S = 5,000$ datasets and for each we compute the semi-parametric score statistic $\hat{S}_{\hat{\gamma}}$ as defined in equation (14) following the Algorithm given in Section 3.[27] We implement the log density score estimator (11) using $B = 4, 6$ or 8 cubic splines and truncate the effective information matrix at machine precision, i.e. $\nu_n^{1/2} = 10^{-308}$.

In Table 2 we show the empirical rejection frequencies under the null corresponding to the $S_{\hat{\gamma}}$ test with nominal level 0.05. The columns correspond to the different choices for the densities ϵ_{ik} for $k \geq 2$. The first column corresponds to the case where all densities are Gaussian and the expected likelihood takes the same value for all $\alpha \in \mathbb{R}^{L_\alpha}$, i.e. α is unidentified. Nonetheless, we find that the empirical rejection frequency of the score test is

[27] To be specific, since the model does not contain any finite dimensional nuisance parameters step 1 in the algorithm can be skipped and the score statistic is simply evaluated at α_0.

always close to the nominal level. This holds regardless of the sample size n, the dimension of the model K and the number of cubic splines B.

Second, when the densities for $k \geq 2$ are non-Gaussian the empirical rejection frequency remains approximately at the nominal level. Specifically, columns 2-4 show the results for the case where ϵ_{ik} follows a Student's t distribution with decreasing degrees of freedom ($\nu = 15, 10, 5$). No matter how close we get to the Gaussian density the empirical rejection frequency remains approximately at the nominal level. Columns 5-10 show that similar properties hold for a variety of mixture distributions. Even for complicated skewed bi-modal densities (e.g. columns 8-10) the $S_{\hat{\gamma}}$ test has empirical rejection frequency close to nominal regardless of the sample size.

Third, overall the number of cubic splines used has little influence on the results. A close inspection reveals that when the number of cubic splines is equal to four the test becomes mildly conservative for some densities, therefore we use $B = 6$ cubic splines in the remaining exercises.

Overall, the asymptotic approximation in Theorem 1 seems to provide a good approximation for the finite sample behavior of the semiparametric score test, at least for the densities shown in Figure 3.

5.2 Comparison to alternative approaches

Next, we compare our semiparametric testing approach to different parametric approaches based on (psuedo) maximum likelihood and the generalized method of moments. We concentrate on evaluating different tests based on their empirical rejection frequency in the vicinity of Gaussianity.[28]

Alternative tests Conceptually, there are two types of alternative tests that we consider: (i) tests that rely on estimates for α and (ii) tests that fix $\alpha = \alpha_0$ under the null. Clearly, from our intuitive discussion in Section 2 it follows that we expect tests that fix α under the null to perform relatively well.

In category (i) we consider the standard maximum likelihood Wald ($\mathrm{W}^{\mathrm{mle}}$) and likelihood ratio ($\mathrm{LR}^{\mathrm{mle}}$) tests based on the Student's t density for ϵ_k. For densities 2-4 in Figure 3 these tests correspond to exact maximum likelihood tests, with the caveat that when the degrees of freedom increases the parameters α become weakly identified, or not-identified. For all other densities these tests are mis-specified.

[28] The recent simulation studies of Herwartz, Lange and Maxand (2019) and Moneta and Pallante (2022) provide further simulation evidence for existing methods, also focusing on estimation accuracy.

In addition, we consider the psuedo-maximum likelihood Wald test ($\mathrm{W}^{\mathrm{pmle}}$) from Gouriéroux, Monfort and Renne (2017). This test is asymptotically valid for a broader range of true distribution functions and amount to fixing the functional form of the densities $\eta_1, \ldots, \eta_K$. We follow the implementation of Gouriéroux, Monfort and Renne (2017) and choose the Students t density with five degrees of freedom as the pseudo-likelihood and compute the Wald statistic based on this density.

Finally, we consider the recently developed GMM method of Lanne and Luoto (2021), which relies on higher order moments to identify the parameters α. We use $\mathbb{E}\epsilon_{ik}^2\epsilon_{ij} = 0$, $\mathbb{E}\epsilon_{ik}^3\epsilon_{ij} = 0$ and $\mathbb{E}\epsilon_{ik}^2\epsilon_{ij}^2 = 1$ as moment conditions for all $j \neq k$ and $j, k = 1, \ldots, K$. The GMM likelihood ratio test is then computed as the rescaled difference between the unrestricted and restricted J-statistics, based on the 2-step GMM estimator ($\mathrm{LR}^{\mathrm{gmm}}$), see Lanne and Luoto (2021) for details.[29]

In category (ii) we consider tests which fix $\alpha = \alpha_0$ under the null. Specifically, we include the standard LM test ($\mathrm{LM}^{\mathrm{mle}}$) based on the Student's t density where the degrees of freedom parameter is estimated from the data. Second, we consider the pseudo-maximum likelihood version of the LM test ($\mathrm{LM}^{\mathrm{pmle}}$) based on Gouriéroux, Monfort and Renne (2017), which fixes the degrees of freedom at five. Finally, we consider the GMM-based identification robust S-statistic ($\mathrm{S}^{\mathrm{gmm}}$) of Stock and Wright (2000), which was recently considered in Drautzburg and Wright (2023) in the context of structural VAR models with non-Gaussian errors. We use the same moment conditions as considered in Drautzburg and Wright (2023) for the $\mathrm{LM}^{\mathrm{gmm}}$ test.

Null rejection frequency comparison We compare the empirical rejection frequencies of the different tests for the simulation designs described in Section 5.1. These are shown in Table 3 for the case where $K = 2$ and $n = 200, 500, 1000$. Overall we find, perhaps not surprisingly, that all tests in category (i) do not demonstrate the correct empirical rejection frequency when the true density is close to Gaussian nor when the corresponding method is based on a mis-specified model. This shows that tests based on estimates for α are generally unreliable. Tests in category (ii) overall demonstrate empirical rejection frequencies close to the nominal level.

More specifically, we find that the Wald tests ($\mathrm{W}^{\mathrm{mle}}$ and $\mathrm{W}^{\mathrm{pmle}}$) tend to over-reject quite severely whilst the standard likelihood ratio test ($\mathrm{LR}^{\mathrm{mle}}$) tends to under-reject for most densities, especially in the vicinity of the Gaussian density, as ought to be expected given the earlier evidence in shown in Figure 1. Finally, the GMM likelihood ratio test ($\mathrm{LR}^{\mathrm{gmm}}$)

[29] Note that lower order moments are not required as the baseline model, $Y_i = A^{-1}\epsilon_i$ with A a rotation matrix, implies that the observations have mean zero and unit variance.

also over-rejects, which confirms findings in Lanne and Luoto (2021) where the $\mathrm{LR}^{\mathrm{gmm}}$ also over-rejects when the densities of the structural shocks are close to Gaussian.

In the second category the semi-parametric score test $\hat{S}_{\hat{\gamma}}$ (as proposed in this paper) and the pseudo maximum likelihood LM test ($\mathrm{LM}^{\mathrm{pmle}}$), inspired by Gouriéroux, Monfort and Renne (2017), both have near perfect empirical rejection frequencies across all densities. The standard LM test ($\mathrm{LM}^{\mathrm{mle}}$) also performs reasonably well, but when the functional form of the true densities is very different from the Student's t density (e.g. separate bi-modal, column 9) the test tends to under-reject.[30] Finally, the GMM based S test ($\mathrm{S}^{\mathrm{gmm}}$) tends to over-reject for small samples, but for large samples it generally shows correct size except for densities with moderately heavy tails such as the $t(5)$ density (column 4). In these cases the $\mathrm{S}^{\mathrm{gmm}}$ over-rejects which can be understood when realizing that the GMM approach requires eight finite moments for inference when based on fourth-order moment restrictions. The $t(5)$ density does not have eight finite moments.

In sum, we recommend avoiding statistics that are based on estimates for α as these are overall unreliable when the shock distributions are close to Gaussian. All tests that fix α under the null perform at least reasonably well.

Power comparison We compare the power of all tests that fix α under the null, that is $\hat{S}_{\hat{\gamma}}$, $\mathrm{LM}^{\mathrm{mle}}$, $\mathrm{LM}^{\mathrm{pmle}}$ and $\mathrm{S}^{\mathrm{gmm}}$.

We consider the case where $K = 2$ and $n = 1000$.[31] In this setting α is a scalar parameter and we fixed the true value at 0 (an arbitrary choice). Figure 4 shows the empirical rejection frequencies when we vary α around $\alpha = 0$. Each point on the curve is based on $S = 5,000$ simulations.

Two main findings stand out. First, for the Student's t densities $t(15)$, $t(10)$ and $t(5)$ (panels 2-4) the standard LM test ($\mathrm{LM}^{\mathrm{mle}}$) shows the highest power. This is not surprising as for these data generating processes the $\mathrm{LM}^{\mathrm{mle}}$ test is correctly specified and hence takes advantage of fitting the true densities using only a scalar parameter. That said, the semi-parametric score test ($\hat{S}_{\hat{\gamma}}$) and the pseudo maximum likelihood LM test ($\mathrm{LM}^{\mathrm{pmle}}$) come reasonably close in terms of power.

Second, for all other densities, i.e. different mixtures of normals in panels $5 - 10$, the semi-parametric score test ($\hat{S}_{\hat{\gamma}}$) shows the highest power. Sometimes the difference with the other tests is not very large, but for instance for bi-modal densities (panels 8-10) the differences are substantial. Overall, the good power of the $\hat{S}_{\hat{\gamma}}$ test corresponds to the theoretical finding that for non-singular information matrices the test is locally asymptotically uniformly most

[30] Recall here that this test is based on a misspecified density.

[31] Power comparisons for different n can be found in the supplementary material.

powerful in the class of (locally asymptotically) unbiased tests.

Besides the $\hat{S}_{\hat{\gamma}}$ test, we note that the pseudo maximum likelihood LM test and the GMM based S test shows quite promising power for most of the densities considered. Neither of these dominates the other. The caveat for the GMM test is that it is size-distorted for moderately heavy tails (panel 4).

5.3 Linear simultaneous equations model

Next, we discuss the simulation results for the general linear simultaneous equations model (3). The dimensions of the design are similar as above with the addition that we consider $d = 2, 3$ for the number of covariates. We now parametrize $A(\alpha, \sigma)^{-1} = \Sigma^{1/2}(\sigma)R(\alpha)$ as in example 3, where $\Sigma^{1/2}$ is lower triangular and the rotation matrix R remains to be specified by the Cayley transform. The explanatory variables are drawn from the standard normal distribution.

The vector of finite dimensional nuisance parameters β now includes $\sigma = \text{vech}(\Sigma^{1/2})$ and $b = \text{vec}(B)$. Our main theoretical result in Theorem 1 permits any $\sqrt{n}$-consistent estimator of β. Obviously, ordinary least squares estimates are attractive for their simplicity, but given the non-normality of the structural shocks these estimators may be improved. Therefore we also consider estimating β by one-step-efficient estimates (e.g. van der Vaart, 2002, Section 7.2), which are easy to compute here since the effective score of β is computed anyway to construct the score test.

Similar to before, the first error $\epsilon_{i,1}$ follows a Gaussian distribution and the different densities from Figure 3 are assigned to the other error terms. For each specification we simulate $S = 5,000$ datasets and for each sample we compute the semi-parametric score statistic using the Algorithm in Section 3.

Null rejection frequency results The empirical rejection frequencies are shown in Tables 4 and 5 for the OLS and one-step efficient estimates for β, respectively.

We find that for all densities the rejection frequencies of the $\hat{S}_{\hat{\gamma}}$ test are generally close to the nominal level. That said, there is more variation in the empirical rejection frequencies compared to Table 2, indicating that the estimation of the finite dimensional nuisance parameters does have consequences.

Starting with Table 4 where $\hat{\beta}$ is estimated by OLS. We find that the empirical rejection frequency of $\hat{S}_{\hat{\gamma}}$ is (approximately) the same regardless of how close the densities of ϵ_{ik} are to the Gaussian density. Specifically, moving from columns 1-4 (i.e. from Gaussian to $t(5)$) we see virtually no changes in the rejection frequencies. This holds for all specifications considered and highlights the main point of this paper: the semi-parametric score test yields reliable inference even when α is not, or poorly, identified.

Depending on the dimension of β we do find distortions in the empirical rejection frequencies for small sample sizes, most notably when $K = 5$ and $n = 200$. In this setting β is of dimension 20 or 25 depending on $d = 2, 3$, and we see that the test often over-rejects. This does not hold for all densities considered, but for Gaussian, Student's t and kurtotic unimodal densities the test over-rejects. When n increases this over-rejection vanishes.

For the one-step efficient estimator for β the results are shown in Table 5. We find that on average the empirical rejection frequencies are larger when compared to the OLS estimator. Notably, when n is small over-rejection becomes more severe. Again, we find that this holds uniformly across all considered densities, i.e. the distortions do not depend on being close to Gaussianity, and the empirical rejection frequencies improve when n increases.

Power results Next, we investigate the power of the $\hat{S}_{\hat{\gamma}}$ test for the LSEM model. We again consider the case where $K = 2$, $d = 2$ and $n = 1000$, which allows us to compare the results with those for the baseline model. The power curves are shown in Figure 5 for both OLS and one-step estimates for β.

First, when comparing Figure 5 to the case without nuisance parameters (i.e. Figure 4) we find that the power of the test is reduced when we include nuisance parameters. Second, the power of the test using the one-step efficient estimates (dotted blue line) is higher when compared to the same test evaluated at OLS estimates. This holds for all densities considered.

Based on these results we recommend using OLS estimates for β when the sample size is small (e.g. $n = 200, 500$), but for larger sample sizes the one-step efficient estimates are preferable.

6 Returns to schooling

In this section, we adopt the semi-parametric score test to construct confidence bands for the effect of education on wages. To do so, we consider a special case of the LSEM model (3): the linear instrumental variable (IV) model, which has been the workhorse model in the returns to schooling literature (e.g. Card, 2001). We show that the presence of non-Gaussian errors allows us to use the score test to (i) obtain tighter confidence bands for the returns to schooling under the assumption that the instrument is exogenous and (ii) test and correct for possibly endogenous instruments.

We start by showing how the standard linear IV model with control variables can be written as a special case of the general model (3). Let y_i be the dependent variable of interest, w_i the scalar endogenous regressor, z_i the $d_z \times 1$ vectors of instruments and X_i the $d \times 1$ vector of control variables. The linear IV model is given by

$$\begin{array}{rcl} y_i & = & \alpha_1 w_i + b_y' X_i + u_i \\ w_i & = & \pi' z_i + b_w' X_i + v_i \\ z_i & = & B_z X_i + e_i \end{array} \quad , \tag{21}$$

where u_i, v_i and e_i are the error terms which are mean zero with variances σ_u^2, σ_v^2 and Σ_e. Further, u_i and v_i are correlated with correlation parameter ρ which captures the endogeneity in the model and prevents us from using basic least squares to estimate α_1. The standard identifying assumption is that e_i is uncorrelated with u_i and v_i such that the instruments given the controls are uncorrelated with the errors.

To write the model in our general notation we first define

$$\begin{bmatrix} u_i \\ v_i \\ e_i \end{bmatrix} = \begin{bmatrix} \sigma_u & 0 & 0 \\ \rho\sigma_v & \sqrt{1-\rho^2}\sigma_v & 0 \\ 0 & 0 & L_e \end{bmatrix} \begin{bmatrix} \epsilon_i^u \\ \epsilon_i^v \\ \epsilon_i^e \end{bmatrix} ,$$

where $\Sigma_e = L_e L_e'$ with L_e lower triangular. To accommodate our general framework we impose that the components of $\epsilon_i = (\epsilon_i^u, \epsilon_i^v, \epsilon_i^e)'$ are mutually independent, with mean zero and unit variance. On this we note that the assumption that the instruments are independent of the error terms u_i and v_i is more commonly imposed (e.g. Hansen, McDonald and Newey, 2010; Cattaneo, Crump and Jansson, 2012), and below we adopt specification tests to assess whether this assumption is reasonable.

Letting $Y_i = (y_i, w_i, z_i')'$ we have

$$Y_i = BX_i + A^{-1}\epsilon_i \ , \qquad \text{where} \quad A^{-1} = \begin{bmatrix} \sigma_u + \alpha_1\sigma_v\rho & \alpha_1\sqrt{1-\rho^2}\sigma_v & \alpha_1\pi' L_e \\ \rho\sigma_v & \sqrt{1-\rho^2}\sigma_v & \pi' L_e \\ 0 & 0 & L_e \end{bmatrix} , \tag{22}$$

and we set $b = \text{vec}(B)$ and $\sigma = (\pi, \sigma_u, \sigma_v, \rho, \text{vech}(L_e)')'$ to summarize the well identified parameters in our general notation. Model (22) is a special case of the LSEM model (3).

The parameter α_1 in the linear IV model may not be identified. The standard requirement is that $\pi \neq 0$. However, the current formulation of the linear IV model shows that with non-Gaussian errors we may be able to locally identify α even when the instruments are irrelevant (e.g. Comon, 1994, Theorem 11). More generally, when the instruments are weak but there is a large degree of non-Gaussianity (relative to sampling variation) we may be able to precisely identify α as the instruments are effectively only used to pin down the desired permutation in A.

We emphasize that Theorem 1 ensures that under weak instrument asymptotics, i.e. $\pi = c/\sqrt{n}$ as in Staiger and Stock (1997), the null rejection probability of the semi-parametric score test for testing $H_0 : \alpha = \alpha_0$ does not exceed the nominal level. At the same time we now have two possible identifying sources: the instruments and the non-Gaussian errors. In this sense the model is over-identified and we use this feature below to test the instrument exogeneity condition.

Data Given this set-up we revisit the returns to schooling problem considered by Card (1995), which uses 1976 wage and schooling data from the 1966 cohort from the NLS to estimate the effect of education on wages. Specifically, for model (22) we set y_i to be the log wage for individual i, w_i is years of eduction, z_i is an indicator for growing up near a 4 year college interacted with parental education and X_i including measures for race, experience, SMSA and region. We refer to Card (2001) for a more general discussion of the literature.

Confidence intervals for the returns to schooling We start by constructing confidence intervals for α_1 in the model (22) by inverting the semi-parametric score test $\hat{S}_{\hat{\gamma}}$ for the null hypothesis $H_0 : \alpha_1 = \alpha_{1,0}$. We compare this approach to inverting the standard the t-statistic for OLS and 2SLS, as well as inverting the weak instrument robust Anderson-Rubin (AR) statistic. The latter does not exploit non-Gaussian errors but has correct null rejection probability under weak instrument asymptotics (e.g. Staiger and Stock, 1997).

Table 6 shows the different confidence intervals together with the point estimates for OLS and 2SLS. We find that the OLS estimate is smaller when compared to the IV estimate and also has a very small confidence interval resonating with the general findings from Card (2001) that OLS is downward biased and having causal estimates presents a cost in terms of accuracy. The 2SLS and AR confidence bands are very similar as the instrument in this application is strong (the effective F-statistic of Montiel Olea and Pflueger (2013) is equal to $F = 80.25$ far exceeding the generalized critical value of 23).

The semi-parametric score test $\hat{S}_{\hat{\gamma}}$ shows the smallest (non − OLS) confidence band for the effect of education on wages $[0.068, 0.105]$, which is considerably smaller when compared to the AR confidence intervals. This reduction in length comes from exploiting non-Gaussian errors in addition to the instrumental variable. Figure 6 shows kernel density estimates for the residuals from the model, i.e. $\hat{\epsilon}_i = \hat{A}\hat{V}_i$, where $\hat{A} = A(\tilde{\alpha}_1, \hat{\sigma})$ with $\tilde{\alpha}_1$ being the value that minimizes the score statistic. We see that there are modest deviations from the Gaussian distribution which are picked up by the score test and explain the shorter length of the confidence interval.

Instrument validity A large part of the discussion in Card (1995) and the subsequent literature is devoted to evaluating the validity of the instruments. Several arguments are presented that question the exogeneity of the proximity to schooling instrument. For instance, the presence of a college may be associated with higher school quality in nearby primary and secondary schools, or with geographical variation in wages. Both are not included in the model specification and hence such associations would invalidate the instrument.

To investigate whether the instruments are indeed invalid we extend the model specification for z_i in (21) to allow for correlation with the error term u_i.

$$z_i = B_z X_i + (\alpha_2/\sigma_u)u_i + e_i \ ,$$

where α_2 captures the correlation of the error term with the instrument. The scaling by σ_u is not necessary but makes the LSEM form below slightly more attractive. When $\alpha_2 = 0$ the instrument is exogenous.

With this extension the LSEM parametrization of the IV model becomes

$$Y_i = BX_i + A^{-1}\epsilon_i \ , \quad A^{-1} = \begin{bmatrix} \sigma_u + \alpha_1\sigma_v\rho + \alpha_1\pi'\alpha_2 & \alpha_1\sqrt{1-\rho^2}\sigma_v & \alpha_1\pi' L_e \\ \rho\sigma_v + \pi'\alpha_2 & \sqrt{1-\rho^2}\sigma_v & \pi' L_e \\ \alpha_2 & 0 & L_e \end{bmatrix} \ , \tag{23}$$

and we test $H_0 : \alpha_1 = \alpha_{1,0}, \alpha_2 = \alpha_{2,0}$ for different values of $\alpha_0 = (\alpha_{1,0}, \alpha_{2,0})$. It is worth pointing out that the inclusion of the additional parameters α_2 prevents the use of standard IV methods, i.e. non-Gaussian errors are needed to distinguish between difference values for α. To do this we use the semi-parametric score test and compare our results to some alternative methods that were discussed in the simulation section.

Figure 7a-(a) shows the joint confidence set for α_1 and α_2 that was obtained by inverting $\hat{S}_{\hat{\gamma}}$. We find that the hypothesis that the instrument is exogenous (i.e. $\alpha_2 = 0$) cannot be rejected, and the 95% confidence set for α_2 is reasonably tight between approximately -0.2 and 0.25. Most importantly, despite relaxing the instrument validity assumption the implied returns to education are very similar: the confidence set indicates with 95% confidence that the effect of education is between 0.06 and 0.12, only a mild increase when compared to the model that assumes instrument exogeneity.

To showcase the advantage of the semi-parametric score test we also computed a confidence set for α by inverting the pseudo maximum likelihood LM test $\mathrm{LM}^{\mathrm{pmle}}$ that was discussed in the simulation study, see Figure 7a-(b). We find that the confidence set is considerably larger in volume.

Specification tests We re-emphasize that the semi-parametric score test was build on the underlying assumption that the components of the errors ϵ_i are independent. For the returns to schooling application this implied the errors ϵ_i^u, ϵ_i^v and ϵ_i^z that determine the structural errors and the instruments are independent. To investigate whether this is a plausible assumption we apply the permutation test for mutual independence as proposed by Matteson and Tsay (2017). The p-value for the test is 0.120 and we may conclude that the independence assumption is not rejected for this application, though the evidence is not overwhelming.

In the supplementary material we consider a more general LSEM model which allows for conditional heteroskedasticity. There we repeated the analyses presented here with the difference that the scalings σ_u, σ_v and L_e are allowed to depend on X_i. We find that resulting confidence set for $\alpha = (\alpha_1, \alpha_2)$ is quite similar when compared to its homoskedastic counterpart.

7 Conclusion

In this paper we highlighted a weak identification problem that can arise when non-Gaussianity is used to identify parameters in LSEMs. The consequence of this problem is that several existing inference methods suffer from size distortions when the true distributions are close to Gaussian.

To reduce this problem we proposed a semi-parametric score statistic for testing hypotheses in LSEMs. Under mild regularity conditions we demonstrated that the semi-parametric score test is locally robust in the sense that its null rejection probability is no greater than the nominal level under parameter sequences that can be described by local deviations from the true parameters which satisfy the null hypothesis (i.e. under weak identification asymptotics). A simulation study shows that our asymptotic theory provides an accurate approximation to the finite sample performance of our test.

While we have restricted our treatment to models where the observations were independently distributed across entities, we note that a similar approach may be considered for dynamic models, but this will require extending our results to allow for non-i.i.d. data. Further, whilst our work shows that the semi-parametric score test is robust under weak identification asymptotics, no global uniformity results are derived. These extensions are left for future work.

References

Amari, Shun-Ichi, and J.-F. Cardoso. 1997. "Blind source separation-semiparametric statistical approach." *IEEE Transactions on Signal Processing*, 45(11): 2692–2700.

Anderson, Theodore W., and Herman Rubin. 1949. "Estimation of the Parameters of a Single Equation in a Complete System of Stochastic Equations." *Annals of Mathematical Statistics*, 20(1): 46–63.

Andrews, Donald W. K. 1987. "Asymptotic Results for Generalized Wald Tests." *Econometric Theory*, 3(3): 348–358.

Andrews, Donald W. K., and Patrik Guggenberger. 2019. "Identification- and singularity-robust inference for moment condition models." *Quantitative Economics*, 10(4): 1703–1746.

Andrews, Donald W. K., and Xu Cheng. 2012. "Estimation and Inference With Weak, Semi-Strong, and Strong Identification." *Econometrica*, 80(5): 2153–2211.

Andrews, Donald W.K., and Xu Cheng. 2013. "Maximum likelihood estimation and uniform inference with sporadic identification failure." *Journal of Econometrics*, 173(1): 36–56.

Andrews, Donald W.K., Xu Cheng, and Patrik Guggenberger. 2020. "Generic results for establishing the asymptotic size of confidence sets and tests." *Journal of Econometrics*, 218(2): 496–531.

Andrews, Isaiah, and Anna Mikusheva. 2015. "Maximum likelihood inference in weakly identified dynamic stochastic general equilibrium models." *Quantitative Economics*, 6(1): 123–152.

Andrews, Isaiah, and Anna Mikusheva. 2022. "Optimal Decision Rules for Weak GMM." *Econometrica*, 90(2): 715–748.

Azzalini, Adelchi, and Antonella Capitanio. 2014. *The Skew-Normal and Related Families.* Cambridge University Press.

Bekaert, Geert, Eric Engstrom, and Andrey Ermolov. 2020. "Aggregate Demand and Aggregate Supply Effects of COVID-19: A Real-time Analysis." *Working paper.*

Bekaert, Geert, Eric Engstrom, and Andrey Ermolov. 2021. "Macro risks and the term structure of interest rates." *Journal of Financial Economics*, 141(2): 479–504.

Bickel, Peter J., Ya'acov Ritov, and Thomas M. Stoker. 2006. "Tailor-made tests for goodness of fit to semiparametric hypotheses." *Annals of Statistics*, 34(2): 721–741.

Bickel, P. J., C. A. J. Klaasen, Y. Ritov, and J. A. Wellner. 1998. *Efficient and Adaptive Estimation for Semiparametric Models.* New York, NY, USA:Springer.

Bonhomme, Stéphane, and Jean-Marc Robin. 2009. "Consistent noisy independent component analysis." *Journal of Econometrics*, 149(1): 12–25.

Card, David. 1995. "Using Geographic Variation in College Proximity to Estimate the Return to Schooling." In *Aspects of Labour Market Behaviour: Essays in Honour of John Vanderkamp.* , ed. Louis N. Christofides, E. Kenneth Grant and Robert Swidinsky, 201–222. University of Toronto Press.

Card, David. 2001. "Estimating the Return to Schooling: Progress on Some Persistent Econometric Problems." *Econometrica*, 69(5): 1127–1160.

Cattaneo, Matias D., Richard K. Crump, and Michael Jansson. 2012. "Optimal inference for instrumental variables regression with non-Gaussian errors." *Journal of Econometrics*, 167(1): 1 – 15.

Chen, A., and P. J. Bickel. 2006. "Efficient Independent Component Analysis." *Annals of Statistics*, 34(6): 2825–2855.

Choi, Sungsub, W. J. Hall, and Anton Schick. 1996. "Asymptotically uniformly most powerful tests in parametric and semiparametric models." *Annals of Statistics*, 24(2): 841–861.

Comon, P. 1994. "Independent component analysis, A new concept?" *Signal Processing*, 36(3): 287–314.

Conway, J. B. 1985. *A course in functional analysis.* New York, NY, USA:Springer.

Dagenais, Marcel G., and Denyse L. Dagenais. 1997. "Higher moment estimators for linear regression models with errors in the variables." *Journal of Econometrics*, 76(1-2): 193–221.

Davis, Richard, and Leon Fernandes. 2022. "Independent Component Analysis with Heavy Tails using Distance Covariance." Working paper.

Davis, Richard, and Serena Ng. 2022. "Time series estimation of the dynamic effects of disaster-type shocks." *Journal of Econometrics.* forthcoming.

de Boor, C. 2001. *A Practical Guide to Splines.* New York, NY, USA:Springer.

Dhrymes, Phoebus J. 1994. *Topics in Advanced Econometrics, Volume II Linear and Nonlinear Simultaneous Equations.* Springer-Verlag New York.

Drautzburg, Thorsten, and Jonathan H. Wright. 2023. "Refining set-identification in VARs through independence." *Journal of Econometrics.* forthcoming.

Erickson, Timothy, and Toni M. Whited. 2000. "Measurement Error and the Relationship between Investment and q." *Journal of Political Economy*, 108(5): 1027–1057.

Erickson, Timothy, and Toni M. Whited. 2002. "Two-step GMM Estimation of the errors-in-variable model using higher order moments." *Econometric Theory*, 18(3): 776–799.

Fiorentini, Gabriele, and Enrique Sentana. 2023. "Discrete mixtures of normals pseudo maximum likelihood estimators of structural vector autoregressions." *Journal of Econometrics*, 235(2): 643–665.

Frisch, R. 1933. "Propagation Problems and Impulse Problems In Dynamic Economics." In *Economic Essays in Honor of Gustav Cassel.* George Allen and Unwin.

Gouriéroux, C., A. Monfort, and J-P. Renne. 2017. "Statistical inference for independent component analysis: Application to structural VAR models." *Journal of Econometrics*, 196: 111–126.

Gouriéroux, Christian, Alain Monfort, and Jean-Paul Renne. 2019. "Identification and Estimation in Non-Fundamental Structural VARMA Models." *The Review of Economic Studies*, 87(4): 1915–1953.

Granziera, Eleonora, Hyungsik Roger Moon, and Frank Schorfheide. 2018. "Inference for VARs identified with sign restrictions." *Quantitative Economics*, 9(3): 1087–1121.

Guay, Alain. 2021. "Identification of structural vector autoregressions through higher unconditional moments." *Journal of Econometrics*, 225(1): 27–46.

Gut, Allan. 2005. *Probability: A Graduate Course. Springer Texts in Statistics*, Springer.

Haavelmo, T. 1943. "The Statistical Implications of a System of Simultaneous Equations." *Econometrica*, 11: 1–12.

Haavelmo, T. 1944. "The Probability Approach in Econometrics." *Econometrica*, 12. Supplement.

Hall, W. J., and David J. Mathiason. 1990. "On Large-Sample Estimation and Testing in Parametric Models." *International Statistical Review*, 58(1): 77–97.

Hansen, Christian, James B. McDonald, and Whitney K. Newey. 2010. "Instrumental Variables Estimation With Flexible Distributions." *Journal of Business & Economic Statistics*, 28(1): 13–25.

Herwartz, Helmut. 2019. "Long-run neutrality of demand shocks: Revisiting Blanchard and Quah (1989) with independent structural shocks." *Journal of Applied Econometrics*, 34(5): 811–819.

Herwartz, Helmut, Alexander Lange, and Simone Maxand. 2019. "Statistical Identification in Svars - Monte Carlo Experiments and a Comparative Assessment of the Role of Economic Uncertainties for the US Business Cycle." CEGE Discussion Paper 375.

Horn, R. A., and C. R. Johnson. 2013. *Matrix Analysis.* . 2 ed., Cambridge University Press.

Hyvärinen, A., J. Karhunen, and E. Oja. 2001. *Independent Component Analysis.* John Wiley & Sons, Inc.

Jin, K. 1992. "Empirical Smoothing Parameter Selection In Adaptive Estimation." *Annals of Statistics*, 20(4): 1844–1874.

Jin, Ze, Benjamin B. Risk, and David S. Matteson. 2019. "Optimization and testing in linear non-Gaussian component analysis." *Statistical Analysis and Data Mining: The ASA Data Science Journal*, 12(3): 141–156.

Kaji, Tetsuya. 2021. "Theory of Weak Identification in Semiparametric Models." *Econometrica*, 89(2): 733–763.

Kapteyn, Arie, and Tom Wansbeek. 1983. "Identification in the Linear Errors in Variables Model." *Econometrica*, 51(6): 1847–1849.

Kleibergen, Frank. 2005. "Testing parameters in GMM without assuming that they are identified." *Econometrica*, 73(4): 1103–1123.

Kocherlakota, S., and K. Kocherlakota. 1991. "Neyman's C(α) test and Rao's efficient score test for composite hypotheses." *Statistics & Probability Letters*, 11(6): 491 – 493.

Lanne, Markku, and Helmut Lütkepohl. 2010. "Structural Vector Autoregressions With Nonnormal Residuals." *Journal of Business & Economic Statistics*, 28(1): 159–168.

Lanne, Markku, and Jani Luoto. 2021. "GMM Estimation of Non-Gaussian Structural Vector Autoregression." *Journal of Business & Economic Statistics*, 39(1): 69–81.

Lanne, M., M. Meitz, and P. Saikkonen. 2017. "Identification and estimation of non-Gaussian structual vector autoregressions." *Journal of Econometrics*, 196: 288–304.

Le Cam, Lucien M. 1960. *Locally Asymptotically Normal Families of Distributions: Certain Approximations to Families of Distributions and Their Use in the Theory of Estimation and Testing Hypotheses. University of California Berkeley, Calif: University of California publications in statistics*, University of California Press.

Le Cam, Lucien M., and Grace L. Yang. 2000. *Asypmtotics in Statistics: Some Basic Concepts.* . 2 ed., New York, NY, USA:Springer.

Lee, Adam. 2023. "Robust and Efficient Inference for Non-Regular Semiparametric Models." Working paper.

Lehmann, Erich L., and Joseph P. Romano. 2005. *Testing Statistical Hypotheses.* . 3rd ed., New York, NY, USA:Springer.

Lewbel, Arthur, Susanne M. Schennach, and Linqi Zhang. 2023. "Identification of a Triangular Two Equation System Without Instruments." *Journal of Business & Economic Statistics*. forthcoming.

Lütkepohl, Helmut, and Maike M. Burda. 1997. "Modified Wald tests under nonregular conditions." *Journal of Econometrics*, 78(2): 315–332.

Magnus, Jan R., Henk G.J. Pijls, and Enrique Sentana. 2021. "The Jacobian of the exponential function." *Journal of Economic Dynamics and Control*, 127: 104–122.

Marron, J. S., and M. P. Wand. 1992. "Exact Mean Integrated Squared Error." *Annals of Statistics*, 20(2): 712–736.

Matteson, David S., and Ruey S. Tsay. 2017. "Independent Component Analysis via Distance Covariance." *Journal of the American Statistical Association*, 112(518): 623–637.

Maxand, Simone. 2018. "Identification of independent structural shocks in the presence of multiple Gaussian components." *Econometrics and Statistics*, 55–68.

Moneta, Alessio, and Gianluca Pallante. 2022. "Identification of Structural VAR Models via Independent Component Analysis: A Performance Evaluation Study." *Journal of Economic Dynamics and Control*, 144. forthcoming.

Moneta, Alessio, Doris Entner, Patrik O. Hoyer, and Alex Coad. 2013. "Causal Inference by Independent Component Analysis: Theory and Applications*." *Oxford Bulletin of Economics and Statistics*, 75(5): 705–730.

Montiel Olea, José Luis, and Carolin Pflueger. 2013. "A Robust Test for Weak Instruments." *Journal of Business & Economic Statistics*, 31(3): 358–369.

Moreira, M. J. 2003. "A conditional likelihood ratio test for structual models." *Econometrica*, 71(4).

Newey, Whitney K. 1990. "Semiparametric efficiency bounds." *Journal of Applied Econometrics*, 5(2): 99–135.

Neyman, Jerzy. 1979. "C(α) Tests and Their Use." *Sankhyā: The Indian Journal of Statistics, Series A (1961-2002)*, 41(1/2): 1–21.

Rao, C. R., and S. K. Mitra. 1971. *Generalized Inverse of Matrices and its Applications.* New York, NY, USA:John Wiley & Sons, Inc.

Reiersøl, Olav. 1950. "Identifiability of a Linear Relation between Variables Which Are Subject to Error." *Econometrica*, 18(4): 375–389.

Risk, Benjamin B., David S. Matteson, and David Ruppert. 2019. "Linear Non-Gaussian Component Analysis Via Maximum Likelihood." *Journal of the American Statistical Association*, 114(525): 332–343.

Rothenberg, Thomas J. 1971. "Identification in Parametric Models." *Econometrica*, 39(3): 577–591.

Rudin, W. 1987. *Real & Complex Analysis.* McGraw Hill.

Rudin, W. 1991. *Functional analysis.* . 2 ed., McGraw Hill, Inc.

Sen, A. 2012. "On the Interrelation Between the Sample Mean and the Sample Variance." *The American Statistician*, 66(2): 112–117.

Sims, Christopher A. 2021. "SVAR Identification through Heteroskedasticity with Misspecified Regimes." working paper.

Staiger, D., and J. H. Stock. 1997. "Instrumental variables regression with weak instruments." *Econometrica*, 65(3): 557–586.

Stock, J. H., and J. H. Wright. 2000. "GMM with weak identification." *Econometrica*, 68(5): 1055–1096.

Strasser, Helmut. 1985. *Mathematical Theory of Statistics: Statistical Experiments and Asymptotic Decision Theory. De Gruyter studies in mathematics*, W. de Gruyter.

Tank, A, E B Fox, and A Shojaie. 2019. "Identifiability and estimation of structural vector autoregressive models for subsampled and mixed-frequency time series." *Biometrika*, 106(2): 433–452.

Tinbergen, Jan. 1939. *Statistical Testing of Business Cycle Theories: Part I: A Method and Its Application to Investment Activity.*

van der Vaart, A. W. 1988. *Statistical Estimation in Large Parameter Spaces. CWI Tracts*, Amsterdam:Centrum voor Wiskunde en Informatica.

van der Vaart, A. W. 1998. *Asymptotic Statistics.* . 1st ed., New York, NY, USA:Cambridge University Press.

van der Vaart, A. W. 2002. "Semiparametric Statistics." In *Lectures on Probability Theory and Statistics: Ecole d'Eté de Probabilités de Saint-Flour XXIX - 1999.* , ed. P. Bernard. Berlin, Germany:Springer.

Velasco, Carlos. 2022. "Identification and Estimation of Structural VARMA Models Using Higher Order Dynamics." *Journal of Business & Economic Statistics.* forthcoming.

Working, E. J. 1927. "What Do Statistical "Demand Curves" Show?" *The Quarterly Journal of Economics*, 41(2): 212–235.

Appendix

In this appendix we provide our main proofs. Regarding notation: $x \coloneqq y$ means that x is defined to be y. The Lebesgue measure on $\mathbb{R}^K$ is denoted by λ_K with $\lambda \coloneqq \lambda_1$ and the standard basis vectors in $\mathbb{R}^K$ are $e_1, \ldots, e_K$. We will make use of the empirical process notation: $Pf \coloneqq \int f \,\mathrm{d}P$, $\mathbb{P}_n f \coloneqq \frac{1}{n}\sum_{i=1}^n f(Y_i)$ and $\mathbb{G}_n f \coloneqq \sqrt{n}(\mathbb{P}_n - P)f$. For any two sequence of probability measures $(Q_n)_{n\in\mathbb{N}}$ and $(P_n)_{n\in\mathbb{N}}$ (where Q_n and P_n are defined on a common measurable space for each $n \in \mathbb{N}$), $Q_n \triangleleft P_n$ indicates that $(Q_n)_{n\in\mathbb{N}}$ is contiguous with respect to $(P_n)_{n\in\mathbb{N}}$. $Q_n \triangleleft\triangleright P_n$ indicates that both $Q_n \triangleleft P_n$ and $P_n \triangleleft Q_n$ hold, see van der Vaart (1998, Section 6.2) for formal definitions. $X \perp\!\!\!\perp Y$ indicates that random vectors X and Y are independent; $X \simeq Y$ indicates that they have the same distribution. $a \lesssim b$ means that a is bounded above by Cb for some constant $C \in (0, \infty)$; the constant C may change from line to line. $\operatorname{cl} X$ means the closure of X. vec^{-1} is the inverse vec operator, i.e. if $b = \operatorname{vec}(B)$ then $B = \operatorname{vec}^{-1}(b)$. If S is a subset of an inner product space $(V, \langle\cdot,\ \cdot\rangle)$, $S^\perp$ is its orthogonal complement, i.e. $S^\perp = \{x \in V : \langle x,\ s\rangle = 0 \text{ for all } s \in S\}$. If $S \subset V$ is complete (hence a Hilbert space) the orthogonal projection of $x \in V$ onto S is $\Pi(x|S)$.

In this appendix and the supplementary material we use notation which explicitly records the dependency of objects on $\theta = (\gamma, \eta)$, including in cases where this was left implicit in the main text to prevent the notation from becoming overly cumbersome. For instance, instead of $A_{k\bullet}$, in the appendices we write $A(\alpha, \sigma)_{k\bullet}$ or $e_k' A(\alpha, \sigma)$.

A Score functions and local asymptotic normality

We first review a number of definitions and establish the semiparametric framework underlying the robust testing approach outlined in this paper.

Formally, the considered model (3) is the collection

$$\mathcal{P}_\Theta = \{P_\theta : \theta \in \Theta\}\ , \tag{24}$$

where each P_θ is the law of the data $W_i = (Y_i, \tilde{X}_i)$ which lies in $\mathcal{W} \subset \mathbb{R}^{K+d-1}$. The parameter space Θ has the form $\Theta = \mathcal{A} \times \mathcal{B} \times \mathcal{H}$, where $\mathcal{A} \subset \mathbb{R}^{L_\alpha}$, $\mathcal{B} \subset \mathbb{R}^{L_\beta}$. $\mathcal{H}$ has the form $\mathscr{Z} \times \prod_{k=1}^K \mathscr{H}$, where $\mathscr{Z}$ is the space of density functions η_0 and $\mathscr{H}$ is the space of density functions η_k such that if $\tilde{X} \sim \eta_0$ and $\epsilon_k \sim \eta_k$ then Assumption 2 parts 1, 3, 4 and 5 hold.[32]

We write a typical element of Θ as $\theta = (\alpha, \beta, \eta)$, where $\beta = (b', \sigma')'$ and it is understood that $\alpha \in \mathcal{A}$, $\beta \in \mathcal{B}$ and $\eta \in \mathcal{H}$. In what follows we will let $V_{\theta,i} \coloneqq Y_i - BX_i$ be the reduced form error so that $A(\alpha, \sigma)V_{\theta,i} = \epsilon_i$. Each P_θ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^{K+d-1}$, with (Lebesgue) density given by

$$p_\theta(W_i) = |\det A(\alpha, \sigma)| \prod_{k=1}^K \eta_k(e_k' A(\alpha, \sigma) V_{\theta,i}) \times \eta_0(\tilde{X}_i)\ , \tag{25}$$

[32] Part 2 of Assumption 2 serves to simplify the form of the effective score function derived in Lemma 3 and is not necessary to set up the model.

and hence log-density

$$\ell_\theta(W_i) = \log|\det A(\alpha,\sigma)| + \sum_{k=1}^{K} \log \eta_k(e_k' A(\alpha,\sigma) V_{\theta,i}) + \log \eta_0(\tilde{X}_i) \ . \tag{26}$$

We now define the scores of model (24) following the definition in van der Vaart (2002).

Definition 1 (Cf. Definition 1.6 in van der Vaart, 2002)**.** *A* differentiable path *is a map $t \mapsto P_t$ from a neighborhood of $0 \in [0,\infty)$ to $\mathcal{P}_\Theta$ such that for some measurable function $s : \mathcal{W} \to \mathbb{R}$,*

$$\int \left[\frac{\sqrt{p_t} - \sqrt{p}}{t} - \frac{1}{2} s \sqrt{p} \right]^2 \mathrm{d}\mu \to 0 \ , \tag{27}$$

as $t \to 0$, where p_t and p respectively denote the densities of P_t and P relative to a σ-finite measure μ. The map $t \to \sqrt{p_t}$ is the root density path and s is the ***score function*** *of the submodel $\{P_t : t \geq 0\}$ at $t = 0$.*

In words, a differentiable path is a one-dimensional parametric submodel $\{P_t : t \geq 0\}$ that is differentiable in quadratic mean at $t = 0$ with score function s. If we let $t \mapsto P_t$ range over a collection of submodels, indexed by $\mathcal{V}$, we will obtain a collection of score functions, say s_j for $j \in \mathcal{V}$.

The differentiable paths we consider have the following form. Let P_t be the measure corresponding to the density with form as in (25) evaluated at $\theta_t := (\gamma + tg, \eta_t)$ where the k-th coordinate of η_t is $\eta_{k,t}^{h_k} := \eta_k(1 + th_k)$ ($k = 0, \ldots, K$), and $(g,h) \in \mathbb{R}^L \times H$, where $H = \prod_{k=0}^{K} H_k$ and each H_k is defined following (6). That such $t \mapsto P_t$ paths are indeed differentiable paths as in Definition 1 is established in the following lemma.

Lemma 1. *Suppose Assumptions 1 and 2 hold and that (α,β) is an interior point of $\mathcal{A} \times \mathcal{B}$. For each $(g,h) \in \mathbb{R}^L \times H := \mathcal{V}$, the map $t \mapsto P_{\theta_t}$ is a differentiable path, with score function $g'\dot{\ell}_\theta + \tilde{h}_0 + \sum_{k=1}^{K} \tilde{h}_k$, where $\dot{\ell}_\theta := \nabla_\gamma \log p_\theta$, $\tilde{h}_0(W) := h_0(\tilde{X})$ and $\tilde{h}_k(W) := h_k(e_k' A(\alpha,\sigma) V_\theta)$. $\dot{\ell}_\theta$ has the form $\dot{\ell}_\theta = (\dot{\ell}_{\theta,\alpha}', \dot{\ell}_{\theta,b}', \dot{\ell}_{\theta,\sigma}')'$, with*

$$\begin{aligned}
\dot{\ell}_{\theta,\alpha,l}(W) &:= \sum_{k=1}^{K} \sum_{j=1, j\neq k}^{K} \zeta_{l,k,j}^{\alpha}(\alpha,\sigma) \phi_k(e_k' A(\alpha,\sigma) V_\theta) e_j' A(\alpha,\sigma) V_\theta \\
&\quad + \sum_{k=1}^{K} \zeta_{l,k,k}^{\alpha}(\alpha,\sigma) [\phi_k(e_k' A(\alpha,\sigma) V_\theta) e_k' A(\alpha,\sigma) V_\theta + 1] \\
\dot{\ell}_{\theta,\sigma,l}(W) &:= \sum_{k=1}^{K} \sum_{j=1, j\neq k}^{K} \zeta_{l,k,j}^{\sigma}(\alpha,\sigma) \phi_k(e_k' A(\alpha,\sigma) V_\theta) e_j' A(\alpha,\sigma) V_\theta \\
&\quad + \sum_{k=1}^{K} \zeta_{l,k,k}^{\sigma}(\alpha,\sigma) [\phi_k(e_k' A(\alpha,\sigma) V_\theta) e_k' A(\alpha,\sigma) V_\theta + 1],
\end{aligned}$$

and

$$\dot{\ell}_{\theta,b}(W)' := -\sum_{k=1}^{K} \phi_k\left(e_k' A(\alpha,\sigma) V_\theta\right) e_k' A(\alpha,\sigma) [X' \otimes I_K].$$

Proof. Let $g = (a, \varrho, s) \in \mathbb{R}^{L_\alpha} \times \mathbb{R}^{L_b} \times \mathbb{R}^{L_\sigma}$. The log density of W under θ_t is then

$$\begin{aligned}
\ell_{\theta_t}(W) &= \log p_{\theta_t}(W) \\
&= \log \eta_0(\tilde{X}) + \log(1 + t h_0(\tilde{X})) + \log|\det(A(\alpha + ta, \sigma + ts))| \\
&\quad + \sum_{k=1}^{K} \log \eta_k \left(e_k' A(\alpha + ta, \sigma + ts)(Y - BX - t \operatorname{vec}^{-1}(\varrho)X)\right) \\
&\quad + \sum_{k=1}^{K} \log \left(1 + t h_k \left(e_k' A(\alpha + ta, \sigma + ts)(Y - BX - t \operatorname{vec}^{-1}(\varrho)X)\right)\right) ,
\end{aligned}$$

By Lemma S6, $t \mapsto \sqrt{p_{\theta_t}}$ is continuously differentiable (pointwise) in a neighbourhood $\mathcal{V}$ of 0. Moreover, if we define $q_t(W) := \frac{\partial \log p_{\theta_x}(W)}{\partial x}\big|_{x=t}$ and $Q_t := P_{\theta_t} q_t(W)^2$, Q_t is finite and continuous in a neighbourhood of 0 by the uniformly integrability of $\{q_t(W)^2 : t \in \mathcal{V}\}$ along with the pointwise continuity of $t \mapsto q_t(W)$, both of which follow from Lemma S6. Hence, by Lemma 1.8 in van der Vaart (2002), $t \mapsto P_{\theta_t}$ is a differentiable path with score function given by the derivative of $\ell_{\theta_t}(W)$ at $t = 0$, which is:

$$\begin{aligned}
&\sum_{k=1}^{K} \phi_k\left(e_k' A(\alpha, \sigma) V_\theta\right) e_k' \sum_{l=1}^{L_\alpha} a_l D_{\alpha,l}(\alpha, \sigma) V_\theta + \sum_{l=1}^{L_\alpha} a_l \operatorname{tr}(A(\alpha, \sigma)^{-1} D_{\alpha,l}(\alpha, \sigma)) \\
&+ \sum_{k=1}^{K} \phi_k\left(e_k' A(\alpha, \sigma) V_\theta\right) e_k' \sum_{l=1}^{L_\sigma} s_l D_{\sigma,l}(\alpha, \sigma) V_\theta + \sum_{l=1}^{L_\sigma} s_l \operatorname{tr}(A(\alpha, \sigma)^{-1} D_{\sigma,l}(\alpha, \sigma)) \\
&- \sum_{k=1}^{K} \phi_k\left(e_k' A(\alpha, \sigma) V_\theta\right) e_k' A(\alpha, \sigma)[X' \otimes I_K]\varrho + h_0(\tilde{X}) + \sum_{k=1}^{K} h_k\left(e_k' A(\alpha, \sigma) V_\theta\right),
\end{aligned} \tag{28}$$

with $D_{x,l}(\alpha, \sigma) = \nabla_{x_l} A(\alpha, \sigma)$ for any $x \in \{\alpha, \sigma\}$ and any l in $\{1, \ldots, L_\alpha\}$ or $\{1, \ldots, L_\sigma\}$ as appropriate. We can re-write the two expressions involving the trace as follows: for any $x \in \{\alpha, \sigma\}$ and appropriate index l we have

$$\begin{aligned}
&\sum_{k=1}^{K} \phi_k(e_k' A(\alpha, \sigma) V_\theta) e_k' D_{x,l}(\alpha, \sigma) V_\theta + \operatorname{tr}(A(\alpha, \sigma)^{-1} D_{x,l}(\alpha, \sigma)) \\
&\quad = \sum_{k=1}^{K} \phi_k(e_k' A(\alpha, \sigma) V_\theta) e_k' D_{x,l}(\alpha, \sigma) A(\alpha, \sigma)^{-1} \epsilon + \operatorname{tr}(D_{x,l}(\alpha, \sigma) A(\alpha, \sigma)^{-1}) \\
&\quad = \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta_{l,k,j}^{x}(\alpha, \sigma) \phi_k(e_k' A(\alpha, \sigma) V_\theta) e_j' A(\alpha, \sigma) V_\theta \\
&\qquad + \sum_{k=1}^{K} \zeta_{l,k,k}^{x}(\alpha, \sigma)[\phi_k(e_k' A(\alpha, \sigma) V_\theta) e_k' A(\alpha, \sigma) V_\theta + 1],
\end{aligned}$$

for $\zeta_{l,k,j}^{x}(\alpha, \sigma) := e_k' D_{x,l}(\alpha, \sigma) A(\alpha, \sigma)^{-1} e_j$. We may therefore write the derivative (28) as

$a'\dot{\ell}_{\theta,\alpha} + \varrho'\dot{\ell}_{\theta,b} + s'\dot{\ell}_{\theta,\sigma} + \dot{\ell}_{\theta,\eta,h}$ where

$$\dot{\ell}_{\theta,\eta,h}(W) := h_0(\tilde{X}) + \sum_{k=1}^{K} h_k\left(e_k' A(\alpha,\sigma)V_\theta\right) = \tilde{h}_0(W) + \sum_{k=1}^{K} \tilde{h}_k(W). \tag{29}$$

An elementary calculation reveals that $g'\dot{\ell}_\theta = a'\dot{\ell}_{\theta,\alpha} + \varrho'\dot{\ell}_{\theta,b} + s'\dot{\ell}_{\theta,\sigma}$. □

As shown in Lemma 1, the score functions corresponding to η are $\dot{\ell}_{\theta,\eta,h}$ as defined in (29), for h ranging over H. These are collected in the set $\mathcal{T}$, as defined in equation (6).

The next Lemma establishes a uniform local asymptotic normality result for (a localised version of) our model. For this we need to specify the notion of convergence on $\mathcal{V} := \mathbb{R}^L \times H$. We equip the product space $\mathcal{V}$ with the norm[33]

$$\|(g,h)\| := \sqrt{\|g\|^2 + \|\tilde{h}_0\|^2_{L_2(P_\theta)} + \sum_{k=1}^{K} \|\tilde{h}_k\|_{L_2(P_\theta)}\}^2} \, .$$

Lemma 2. *Suppose that Assumptions 1 and 2 hold and that (α,β) is an interior point of $\mathcal{A} \times \mathcal{B}$. For $(g,h) \in \mathcal{V}$ let*

$$\theta_n(g,h) := \theta + n^{-1/2}(g, \eta_0 h_0, \ldots, \eta_K h_K).$$

For any convergent sequence $(g_n,h_n) \to (g,h)$ (all in $\mathcal{V}$), define R_n as

$$R_n := \log \prod_{i=1}^{n} \frac{p_{\theta_n(g_n,h_n)}(W_i)}{p_\theta(W_i)} - \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[g'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K} \tilde{h}_k(W_i) \right] + \frac{1}{2}\mathbb{E}\left[g'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K} \tilde{h}_k(W_i) \right]^2 .$$

Then,

1. *$R_n \xrightarrow{P_\theta} 0$,*
2. *Under P_θ,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[g'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K} \tilde{h}_k(W_i) \right] \rightsquigarrow \mathcal{N}\left(0, \mathbb{E}\left[g'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K} \tilde{h}_k(W_i) \right]^2 \right),$$

3. *The (product) measures $P^n_{\theta_n}$ and P^n_θ are mutually contiguous.*

Proof. Part 2 follows from Lemma 1 in combination with the Lindenberg-Lévy central limit theorem and Lemma 1.7 of van der Vaart (2002). For Part 1, we first note that in the special case where $(g_n,h_n) = (g,h)$ for all $n \in \mathbb{N}$, $R_n \xrightarrow{P_\theta} 0$ follows by combining Lemma 1 with Lemma 1.9 in van der Vaart (2002). For the general case, note that by Lemma S7 (i) the functions $(g,h) \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[g'\dot{\ell}_\theta + \sum_{k=0}^{K} \tilde{h}_k \right]$ (i.e. indexed by n) are equicontinuous

[33] Each $\tilde{h}_k$ is as defined in the statement of Lemma 1.

on compacts in $L_2(P_\theta)$ and (ii) the functions $(g,h) \mapsto P^n_{\theta_n(g,h)}$ (i.e. indexed by n) are equicontinuous on compacts in the total variation metric. By (i), the i.i.d. assumption and Lemma 1.7 in van der Vaart (2002)

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[(g-g_n)'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K}\left(\tilde{h}_k(W_i) - \tilde{h}_{n,k}(W_i)\right)\right]^2 \\
&= \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[(g-g_n)'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K}\left(\tilde{h}_k(W_i) - \tilde{h}_{n,k}(W_i)\right)\right]\right]^2 \\
&= 0.
\end{aligned} \tag{30}$$

By (ii) one has $\lim_{n\to\infty} d_{TV}(P^n_{\theta_n(g_n,h_n)}, P^n_{\theta_n(g,h)}) = 0$ where d_{TV} indicates the total variation metric. This implies (cf. Theorem 80.13 in Strasser (1985))

$$\log \prod_{i=1}^{n} \frac{p_{\theta_n(g_n,h_n)}(W_i)}{p_\theta(W_i)} - \log \prod_{i=1}^{n} \frac{p_{\theta_n(g,h)}(W_i)}{p_\theta(W_i)} = o_{P^n_\theta}(1).$$

Combine the preceding two displays with the previously demonstrated result for the special case where $(g_n, h_n) = (g,h)$ for all $n \in \mathbb{N}$ to conclude. Part 3 then follows by combining Parts 1 and 2 with Example 6.5 in van der Vaart (1998). □

B Orthogonality and the effective score

We now derive the effective score for α, i.e. $\tilde{\kappa}_\theta$. By definition, this is the orthogonal projection of the score function for the parameter of interest, i.e. $\dot{\ell}_{\theta,\alpha}$, on the orthocomplement (in $L_2(P_\theta)$) of the space spanned by the score functions for all nuisance parameters, i.e. $\dot{\ell}_{\theta,\sigma}$, $\dot{\ell}_{\theta,b}$ and $\dot{\ell}_{\theta,\eta,h}$.[34] That is, collecting the scores for the nuisance parameters as

$$\mathcal{S} := \mathrm{Span}(\dot{\ell}_{\theta,b}) + \mathrm{Span}(\dot{\ell}_{\theta,\sigma}) + \mathcal{T} \subset L_2(P_\theta),$$

where $\mathcal{T}$ is defined in (6) and collects the scores corresponding to η, one has

$$\tilde{\kappa}_{\theta,l} := \Pi\left(\dot{\ell}_{\theta,\alpha,l}\middle|\mathcal{S}^\perp\right),$$

for each $l = 1, \ldots, L_\alpha$.

It is convenient to calculate this projection in two steps (see Bickel et al., 1998, p. 74). Firstly we calculate the effective score for the Euclidean parameters γ, i.e. the orthogonal projection of $(\dot{\ell}'_{\theta,\alpha}, \dot{\ell}'_{\theta,\sigma}, \dot{\ell}'_{\theta,b})'$ onto the orthocomplement of the space spanned by the score functions for the infinite dimensional parameter η, i.e. $\mathcal{T}^\perp$. We denote this by

[34] The terminology "effective score" is taken from Choi, Hall and Schick (1996); much of the semiparametric literature calls this object the "efficient score" (e.g. Bickel et al., 1998; van der Vaart, 1998).

$\tilde{\ell}_\theta \coloneqq (\tilde{\ell}'_{\theta,\alpha}, \tilde{\ell}'_{\theta,\sigma}, \tilde{\ell}'_{\theta,b})' = (\tilde{\ell}'_{\theta,\alpha}, \tilde{\ell}'_{\theta,\beta})'$, i.e. for any $x \in \{\alpha, \sigma, b\}$ and l in $\{1, \ldots, L_x\}$

$$\tilde{\ell}_{\theta,x,l} = \Pi\left(\dot{\ell}_{\theta,x,l} \middle| \mathcal{T}^\perp\right) . \tag{31}$$

For the second step, we may partition

$$\tilde{\ell}_\theta = \left(\tilde{\ell}'_{\theta,\alpha}, \tilde{\ell}'_{\theta,\beta}\right)' \quad \text{and} \quad \tilde{I}_\theta = \begin{bmatrix} \tilde{I}_{\theta,\alpha\alpha} & \tilde{I}_{\theta,\alpha\beta} \\ \tilde{I}_{\theta,\beta\alpha} & \tilde{I}_{\theta,\beta\beta} \end{bmatrix} , \tag{32}$$

with $\tilde{I}_\theta \coloneqq P_\theta[\tilde{\ell}_\theta \tilde{\ell}'_\theta]$. If $\tilde{I}_{\theta,\beta\beta}$ is nonsingular,[35] we can (orthogonally) project once more to obtain the effective score function for α:[36]

$$\tilde{\varkappa}_\theta = \tilde{\ell}_{\theta,\alpha} - \tilde{I}_{\theta,\alpha\beta} \tilde{I}^{-1}_{\theta,\beta\beta} \tilde{\ell}_{\theta,\beta} , \tag{33}$$

which has corresponding effective information matrix

$$\tilde{\mathcal{I}}_\theta \coloneqq \tilde{I}_{\theta,\alpha\alpha} - \tilde{I}_{\theta,\alpha\beta} \tilde{I}^{-1}_{\theta,\beta\beta} \tilde{I}_{\theta,\beta\alpha} . \tag{34}$$

Lemma 3. *Suppose Assumptions 1 and 2 hold. Then the components of $\tilde{\ell}_\theta$ are as follows. For $x = \alpha$ or $x = \sigma$,*

$$\begin{aligned} \tilde{\ell}_{\theta,x,l}(W) = & \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \zeta^x_{l,k,j}(\alpha,\sigma) \phi_k(e'_k A(\alpha,\sigma) V_\theta) e'_j A(\alpha,\sigma) V_\theta \\ & + \sum_{k=1}^{K} \zeta^x_{l,k,k}(\alpha,\sigma)(\tau_{k,1} e'_k A(\alpha,\sigma) V_\theta + \tau_{k,2} \varkappa(e'_k A(\alpha,\sigma) V_\theta)), \end{aligned}$$

with l in $\{1, \ldots, L_\alpha\}$ or $\{1, \ldots, L_\sigma\}$ (respectively); for $x = b$,

$$\begin{aligned} \tilde{\ell}_{\theta,b}(W) = & -\sum_{k=1}^{K} \phi_k(e'_k A(\alpha,\sigma) V_\theta) e'_k A(\alpha,\sigma) \left([X' \otimes I_K] - \mathbb{E}[(X' \otimes I_K)]\right) \\ & + \sum_{k=1}^{K} e'_k A(\alpha,\sigma) \mathbb{E}[(X' \otimes I_K)](\varsigma_{k,1} e'_k A(\alpha,\sigma) V_\theta + \varsigma_{k,2} \varkappa(e'_k A(\alpha,\sigma) V_\theta)); \end{aligned}$$

[35] If $\tilde{I}_{\theta,\beta\beta}$ is singular, we may drop components from $\tilde{\ell}_{\theta,\beta}$ until the remaining components form a linearly independent collection which span the same subspace of $L_2(P_\theta)$ as $\ell_{\theta,\beta}$. The corresponding variance matrix of this smaller vector will be non-singular and $\tilde{\ell}_{\theta,\beta}$ can be replaced throughout by this smaller vector.

[36] For any $l = 1, \ldots, L_\alpha$, one has that

$$\varkappa_{\theta,l} = \tilde{\ell}_{\theta,\alpha,l} - e'_l \tilde{I}_{\theta,\alpha\beta} \tilde{I}^{-1}_{\theta,\beta\beta} \tilde{\ell}_{\theta,\beta} = \Pi\left(\tilde{\ell}_{\theta,\alpha,l} \middle| \left[\operatorname{Span}\left(\tilde{\ell}_{\theta,\beta}\right)\right]^\perp\right) .$$

where the expectations are taken under P_θ and

$$\tau_k := M_k^{-1}\begin{pmatrix}0\\-2\end{pmatrix},\ \varsigma_k := M_k^{-1}\begin{pmatrix}1\\0\end{pmatrix},\ \textit{for } M_k := \begin{pmatrix}1 & \mathbb{E}[\epsilon_k^3]\\ \mathbb{E}[\epsilon_k^3] & \mathbb{E}[\epsilon_k^4]-1\end{pmatrix}.$$

Proof. For each $h_k \in H_k$, define the corresponding $\tilde{h}_k$ as in the statement of Lemma 1 and let $\tilde{H}_k$ collect all such $\tilde{h}_k$ formed with h_k ranging over H_k.[37] By the definition of $\tilde{\ell}_\theta$ in equation (31) and Theorem 4.11 in Rudin (1987) it suffices to show that each such component is (a) in $(\tilde{H}_0 + \cdots + \tilde{H}_K)^\perp$ and (b) $\dot{\ell}_{\theta,x} - \tilde{\ell}_{\theta,x} \in \mathrm{cl}(\tilde{H}_0 + \cdots + \tilde{H}_K)$, the form of which is given in Lemma S8.

Case 1: $x = \alpha, \sigma$. For (a) note that if $j \neq k$, then

$$\begin{aligned}\mathbb{E}\left[\zeta_{l,k,j}^x(\alpha,\sigma)\phi_k(\epsilon_k)\epsilon_j h_0(\tilde{X})\right] &= \mathbb{E}\left[\zeta_{l,k,j}^x(\alpha,\sigma)\phi_k(\epsilon_k)h_0(\tilde{X})\right]\mathbb{E}[\epsilon_j] = 0\\ \mathbb{E}\left[\zeta_{l,k,j}^x(\alpha,\sigma)\phi_k(\epsilon_k)\epsilon_j h_m(\epsilon_m)\right] &= \mathbb{E}\left[\zeta_{l,k,j}^x(\alpha,\sigma)\right]\mathbb{E}\left[\phi_k(\epsilon_k)\epsilon_j h_m(\epsilon_m)\right] = 0\end{aligned}$$

where the last equality follows from independence and the fact that m must differ from one of k, j. Additionally, by independence and our moment assumptions (i.e. Assumption 2)

$$\mathbb{E}\left[\left(\zeta_{l,k,j}^x(\alpha,\sigma)[\tau_{k,1}\epsilon_k + \tau_{k,2}\kappa(\epsilon_k)]\right)h_0(\tilde{X})\right] = \zeta_{l,k,j}^x(\alpha,\sigma)\mathbb{E}\left[\tau_{k,1}\epsilon_k + \tau_{k,2}\kappa(\epsilon_k)\right]\mathbb{E}[h_0(\tilde{X})] = 0,$$

and again using independence and the definition of H_k,

$$\mathbb{E}\left[\zeta_{l,k,j}^x(\alpha,\sigma)[\tau_{k,1}\epsilon_k + \tau_{k,2}\kappa(\epsilon_k)]h_j(\epsilon_j)\right] = \zeta_{l,k,j}^x(\alpha,\sigma)\mathbb{E}\left[\left(\tau_{k,1}\epsilon_k + \tau_{k,2}\kappa(\epsilon_k)\right)h_j(\epsilon_j)\right] = 0.$$

Since $\epsilon_k = e_k' A(\alpha,\sigma)V_\theta$, these observations and the form of $\tilde{\ell}_{\theta,x}$ establish (a). For (b), it suffices to show that

$$f_k(\epsilon_k) := \phi_k(\epsilon_k)\epsilon_k + 1 - \tau_{k,1}\epsilon_k - \tau_{k,2}\kappa(\epsilon_k) \in H_k.$$

That $\mathbb{E}[f_k(\epsilon_k)] = 0$ and $\mathbb{E}[f_k(\epsilon_k)^2] < \infty$ follows immediately from Assumption 2. That additionally $\mathbb{E}[f_k(\epsilon_k)\epsilon_k] = \mathbb{E}[f_k(\epsilon_k)\kappa(\epsilon_k)] = 0$ is ensured by the choice of τ_k.

Case 2: $x = b$. For (a) let $m(X) := A(\alpha,\sigma)(X' \otimes I_K)$ and $\mu = \mathbb{E}[m(X)]$. Then,

$$\begin{aligned}\mathbb{E}[\phi_k(\epsilon_k)e_k'(m(X)-\mu)h_0(\tilde{X})] &= \mathbb{E}[\phi_k(\epsilon_k)]\mathbb{E}[e_k'(m(X)-\mu)h_0(\tilde{X})] = 0\\ \mathbb{E}[\phi_k(\epsilon_k)e_k'(m(X)-\mu)h_j(\epsilon_j)] &= \mathbb{E}[\phi_k(\epsilon_k)h_j(\epsilon_j)]\mathbb{E}[e_k'(m(X)-\mu)] = 0\\ \mathbb{E}[e_k'\mu\left(\varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)\right)h_0(\tilde{X})] &= e_k'\mu\mathbb{E}[\varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)]\mathbb{E}[h_0(\tilde{X})] = 0;\end{aligned}$$

for $k \neq j$ by independence

$$\mathbb{E}[e_k'\mu\left(\varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)\right)h_j(\epsilon_j)] = e_k'\mu\mathbb{E}[\varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)]\mathbb{E}[h_j(\epsilon_j)] = 0$$

[37] That is, for each $h_0 \in H_0$ define $\tilde{h}_0 : \mathcal{W} \to \mathbb{R}$ acccording to $\tilde{h}_0(W) := h_0(\tilde{X})$ and let $\tilde{H}_0$ collect the $\tilde{h}_0$ functions so formed. Similarly, for each $h_k \in H_k$ $(k = 1, \ldots, K)$, define $\tilde{h}_k : \mathcal{W} \to \mathbb{R}$ according to $\tilde{h}_k(W) := h_k(e_k' A(\alpha,\sigma)V_\theta)$ and let let $\tilde{H}_k$ collect the $\tilde{h}_k$ functions so formed.

whilst for $k = j$, the definition of H_k ensures that

$$\mathbb{E}[e_k'\mu\left(\varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)\right)h_k(\epsilon_k)] = e_k'\mu\mathbb{E}[\varsigma_{k,1}\epsilon_k h_k(\epsilon_k) + \varsigma_{k,2}\kappa(\epsilon_k)h_k(\epsilon_k)] = 0.$$

Since $\epsilon_k = e_k'A(\alpha,\sigma)V_\theta$, these observations and the form of $\tilde{\ell}_{\theta,b}$ establish (a). For (b) it suffices to show that

$$q_k(\epsilon_k) \coloneqq \left(\phi_k(\epsilon_k) + \varsigma_{k,1}\epsilon_k + \varsigma_{k,2}\kappa(\epsilon_k)\right)(-e_k'\mu) \in H_k.$$

That $\mathbb{E}[q_k(\epsilon_k)] = 0$ and $\mathbb{E}[q_k(\epsilon_k)^2] < \infty$ follows immediately from Assumption 2. That additionally $\mathbb{E}[q_k(\epsilon_k)\epsilon_k] = \mathbb{E}[q_k(\epsilon_k)\kappa(\epsilon_k)] = 0$ is ensured by the choice of ς_k. □

C Proof of Theorem 1

C.1 Log density score estimation

As discussed just prior to Assumption 3, the log density score estimator in (11) may be replaced by an alternative estimator, provided it satisfies some high level conditions. These are given in the following assumption.

Assumption 4. *Let ν_n be as in Assumption 3. We have estimators $\hat{\phi}_{k,n,\gamma}$ such that for (a) any sequence with elements $\theta_n = (\alpha_0, \beta_n, \eta) \in \Theta$ where $(\beta_n)_{n\in\mathbb{N}}$ is a deterministic sequence with $\sqrt{n}\|\beta_n - \beta\| = O(1)$ and (b) any array $(Z_{n,i})_{n\in\mathbb{N},i\leq n}$ with i.i.d. rows and such that $\mathbb{E}Z_{n,i} = 0$, $\sup_{n\in\mathbb{N}}\mathbb{E}Z_{n,i}^2 < \infty$ and $Z_{n,i} \perp\!\!\!\perp \epsilon_{i,k}$ for each n, i, and k,*

$$\frac{1}{n}\sum_{i=1}^{n}\left[\hat{\phi}_{k,n,\gamma_n}(A_{k,\gamma_n}V_{\theta_n,i}) - \phi_k(A_{k,\gamma_n}V_{\theta_n,i})\right]Z_{n,i} = o_{P_{\theta_n}^n}(n^{-1/2}), \tag{35}$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\left[\hat{\phi}_{k,n,\gamma_n}(A_{k,\gamma_n}V_{\theta_n,i}) - \phi_k(A_{k,\gamma_n}V_{\theta_n,i})\right]Z_{n,i}\right)^2 = o_{P_{\theta_n}^n}(\nu_n). \tag{36}$$

where $A_{k,\gamma_n} \coloneqq e_k'A(\alpha_0,\sigma_n)$, $V_{\theta_n,i} \coloneqq Y_i - \mathrm{vec}^{-1}(b_n)X_i$.

The following Lemma verifies that, under Assumptions 2 and 3, the log density score estimator in (11) satisfies Assumption 4. Its proof is given in Section S5 of the supplementary appendix.

Lemma 4. *Suppose Assumptions 2 and 3 hold. Then, $\hat{\phi}_{k,n,\gamma} \coloneqq \hat{\phi}_{k,n}$ as defined in (11) satisfies Assumption 4.*

C.2 Proof of Theorem 1

In order to prove Theorem 1, we first establish two results which give high level conditions under which Theorem 1 holds. The proof of Theorem 1 then consists of verifying the required high level conditions under our primitive assumptions. Let us first recall the definitions of various objects which were introduced in Section 3.

We have that $\tilde{\ell}_\theta$ denotes the *effective score* for Euclidean parameter vector $\gamma = (\alpha, \beta)$, evaluated at θ (as defined in (31) and derived in Lemma 3). The *effective information* for γ is denoted $\tilde{I}_\theta := P_\theta[\tilde{\ell}_\theta \tilde{\ell}_\theta']$. Given a $\gamma = (\alpha, \beta)$, these objects are estimated by $\hat{\ell}_{n,\gamma} = \hat{\ell}_{n,\gamma}(W_1, \ldots, W_n)$ and $\hat{I}_{n,\gamma} = \hat{I}_{n,\gamma}(W_1, \ldots, W_n)$, respectively. Each of these objects can be partitioned conformally with (α, β):

$$\tilde{\ell}_\theta = \begin{pmatrix} \tilde{\ell}_{\theta,\alpha} \\ \tilde{\ell}_{\theta,\beta} \end{pmatrix}, \; \hat{\ell}_{n,\gamma} = \begin{pmatrix} \hat{\ell}_{n,\gamma,\alpha} \\ \hat{\ell}_{n,\gamma,\beta} \end{pmatrix}, \; \tilde{I}_\theta = \begin{pmatrix} \tilde{I}_{\theta,\alpha\alpha} & \tilde{I}_{\theta,\alpha\beta} \\ \tilde{I}_{\theta,\beta\alpha} & \tilde{I}_{\theta,\beta\beta} \end{pmatrix}, \; \text{ and } \hat{I}_{n,\gamma} = \begin{pmatrix} \hat{I}_{n,\gamma,\alpha\alpha} & \hat{I}_{n,\gamma,\alpha\beta} \\ \hat{I}_{n,\gamma,\beta\alpha} & \hat{I}_{n,\gamma,\beta\beta} \end{pmatrix}.$$

The effective score for α is $\tilde{\kappa}_\theta := \tilde{\ell}_{\theta,\alpha} - \tilde{I}_{\theta,\alpha\beta}\tilde{I}_{\theta,\beta\beta}^{-1}\tilde{\ell}_{\theta,\beta}$, with corresponding effective information $\tilde{\mathcal{I}}_\theta := \tilde{I}_{\theta,\alpha\alpha} - \tilde{I}_{\theta,\alpha\beta}\tilde{I}_{\theta,\beta\beta}^{-1}\tilde{I}_{\theta,\beta\alpha}$.[38] For a given γ, the estimator of $\tilde{\kappa}_\theta$ is

$$\hat{\kappa}_{n,\gamma} := \hat{\ell}_{n,\gamma,\alpha} - \hat{I}_{n,\gamma,\alpha\beta}\hat{I}_{n,\gamma,\beta\beta}^{-1}\hat{\ell}_{n,\gamma,\beta}.$$

The estimator of the effective information for α, $\tilde{\mathcal{I}}_\theta$, is formed in two steps. Firstly, the preliminary estimate $\check{\mathcal{I}}_{n,\gamma} := \hat{I}_{n,\gamma,\alpha\alpha} - \hat{I}_{n,\gamma,\alpha\beta}\hat{I}_{n,\gamma,\beta\beta}^{-1}\hat{I}_{n,\gamma,\beta\beta}$ is formed by replacing population quantities by their sample equivalents. Secondly, the regularized estimator $\hat{\mathcal{I}}_{n,\gamma}$ is formed as in (15): let $\check{U}_{n,\gamma}\check{\Lambda}_{n,\gamma}\check{U}_{n,\gamma}'$ be the eigendecomposition of the initial estimator $\check{\mathcal{I}}_{n,\gamma}$. $\check{\Lambda}_{n,\gamma}$ is a diagonal matrix with (i,i)th element $\check{\lambda}_{n,\gamma,i}$. Then the estimator is:

$$\hat{\mathcal{I}}_{n,\gamma}^t = \check{U}_{n,\gamma}\hat{\Lambda}_{n,\gamma}(\nu_n^{1/2})\check{U}_{n,\gamma}' \; ,$$

where $\hat{\Lambda}_{n,\gamma}(\nu_n^{1/2})$ is a diagonal matrix with the $\nu_n^{1/2}$-truncated eigenvalues of $\hat{\mathcal{I}}_{n,\gamma}$ on the main diagonal, i.e. the (i,i)–th element of $\hat{\Lambda}_n(\nu_n^{1/2})$ is $\mathbf{1}(\check{\lambda}_{n,\gamma,i} \geq \nu_n^{1/2})$. The rank estimator used is $\hat{r}_{n,\gamma} = \text{rank}(\hat{\mathcal{I}}_{n,\gamma}^t)$. Finally, the effective score statistic (for a given γ) is given by

$$\hat{S}_{n,\gamma} := n\left(\mathbb{P}_n\hat{\kappa}_{n,\gamma}\right)'\hat{\mathcal{I}}_{n,\gamma}^{t,\dagger}\left(\mathbb{P}_n\hat{\kappa}_{n,\gamma}\right),$$

where $\hat{\mathcal{I}}_{n,\gamma}^{t,\dagger}$ is the Moore – Penrose psuedoinverse of $\hat{\mathcal{I}}_{n,\gamma}^t$.

Theorem 2. *Suppose that for any deterministic sequence $(\tilde{\theta}_n)_{n\in\mathbb{N}}$ in Θ with elements $\tilde{\theta}_n = (\alpha, \beta_n, \eta)$ such that $\sqrt{n}\|\beta_n - \beta\| = O(1)$ the following conditions hold:*

1. *The functions $\tilde{\ell}_{\theta_n}$ satisfy*
$$\sqrt{n}\mathbb{P}_n\left[\tilde{\ell}_{\tilde{\theta}_n} - \tilde{\ell}_\theta\right] + \sqrt{n}\tilde{I}_\theta\begin{pmatrix} 0 \\ \beta_n - \beta \end{pmatrix} = o_{P_\theta^n}(1); \tag{37}$$
2. *The estimators $\hat{\ell}_{n,\gamma_n}$ satisfy $\sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right] = o_{P_{\tilde{\theta}_n}^n}(1)$;*
3. *The estimators $\hat{I}_{n,\gamma_n}$ satisfy $\|\hat{I}_{n,\gamma_n} - \tilde{I}_\theta\|_2 = o_{P_{\tilde{\theta}_n}^n}(\nu_n^{1/2})$ for a non-negative sequence $(\nu_n)_{n\in\mathbb{N}}$ with $\nu_n \to 0$;*

[38] Here it is assumed that $\tilde{I}_{\theta,\beta\beta}$ is non-singular; cf. footnote 35.

where $\gamma_n \coloneqq (\alpha, \beta_n)$, $\theta \coloneqq (\alpha, \beta, \eta)$ and $\tilde{I}_\theta \coloneqq P_\theta[\tilde{\ell}_\theta \tilde{\ell}_\theta']$. Moreover, suppose that for $(g_n, h_n) \to (g, h)$ (all in $\mathcal{V}$) and some $\sigma(g,h) \in (0, \infty)$, under P_θ^n

$$\left(\sqrt{n}\mathbb{P}_n\tilde{\ell}_\theta,\ \log\prod_{i=1}^n \frac{p_{\theta_n(g_n,h_n)}}{p_\theta}\right) \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} 0 \\ -\frac{1}{2}\sigma(g,h)\end{pmatrix}, \begin{pmatrix} \tilde{I}_\theta & \tilde{I}_\theta g \\ g'\tilde{I}_\theta & \sigma(g,h)\end{pmatrix}\right), \tag{38}$$

where $\theta_n(g,h)$ is as in Lemma 2. Suppose that initial estimators $\hat{\beta}_n$ are available with $\sqrt{n}\|\hat{\beta}_n - \beta\| = O_{P_\theta^n}(1)$ and let $\bar{\beta}_n$ be a discretised version of this which takes values in $\mathsf{G}_n \coloneqq n^{-1/2}C\mathbb{Z}^{L_\beta}$ for some $C \in (0, \infty)$.[39] *Then, if $\bar{\gamma}_n \coloneqq (\alpha, \bar{\beta}_n)$ and $r \coloneqq \operatorname{rank}\tilde{\mathcal{I}}_\theta$,*

$$\sqrt{n}\mathbb{P}_n\hat{\kappa}_{n,\bar{\gamma}_n} = \sqrt{n}\mathbb{P}_n\tilde{\kappa}_\theta + o_{P^n_{\theta_n(g_n,h_n)}}(1) \rightsquigarrow \mathcal{N}(0, \tilde{\mathcal{I}}_\theta), \qquad \textit{and} \qquad \hat{S}_{n,\bar{\gamma}_n} \rightsquigarrow \chi^2_r, \tag{39}$$

under any $P^n_{\theta_n(g_n,h_n)}$ such that $(g_n, h_n) \to (g, h)$ (all in $\mathcal{V}$) with $g = (0, (b, s)')' \in \mathbb{R}^{L_\alpha} \times \mathbb{R}^{L_\beta}$. Additionally, under any $P^n_{\theta_n(g_n,h_n)}$ such that $(g_n, h_n) \to (g, h)$ (all in $\mathcal{V}$),

$$\hat{r}_{n,\bar{\gamma}_n} \xrightarrow{P^n_{\theta_n(g_n,h_n)}} r. \tag{40}$$

Proof. Step 1: Let $d_n \coloneqq \sqrt{n}(\beta_n - \beta)$. By arguing along subsequences if necessary we may assume without loss of generality that $d_n \to d$. Hence for $g_n^\diamond \coloneqq (0, d_n')' \to (0, d')' =: g^\diamond$, $\tilde{\theta}_n = \theta_n(g_n^\diamond, 0)$. By condition (38) and Example 6.5 in van der Vaart (1998), $P^n_{\tilde{\theta}_n} \lhd\rhd P^n_\theta$ and so, given the assumed convergences in conditions 2 and 3, we have

$$\sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right] = o_{P_\theta^n}(1) \quad \text{and} \quad \|\hat{I}_{n,\gamma_n} - \tilde{I}_\theta\|_2 = o_{P_\theta^n}(\nu_n^{1/2}).$$

Step 2: We show that the convergences in the preceding display and equation (37) continue to hold if γ_n (and $\theta_n = (\gamma_n, \eta)$) is replaced by $\bar{\gamma}_n$ (and $\bar{\theta}_n = (\bar{\gamma}_n, \eta)$) as in the statement of the theorem. Let $\gamma^\star = (\alpha, \beta^\star)$ and $\theta^\star = (\gamma^\star, \eta)$ and define

$$\begin{aligned} R_{n,1}(\beta^\star) &\coloneqq \sqrt{n}\mathbb{P}_n\left[\tilde{\ell}_{\theta^\star} - \tilde{\ell}_\theta\right] + \sqrt{n}\tilde{I}_\theta\begin{pmatrix} 0 \\ \beta^\star - \beta\end{pmatrix} \\ R_{n,2}(\beta^\star) &\coloneqq \sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\gamma^\star} - \tilde{\ell}_{\theta^\star}\right] \\ R_{n,3}(\beta^\star) &\coloneqq \nu_n^{-1/2}\left[\hat{I}_{n,\gamma^\star} - \tilde{I}_\theta\right]. \end{aligned}$$

For any $\varepsilon > 0$ there is an M such that $P_\theta^n(\sqrt{n}\|\hat{\beta}_n - \beta\| > M) < \epsilon$. Moreover, whenever $\sqrt{n}\|\hat{\beta}_n - \beta\| \le M$ then $\bar{\beta}_n \in \mathsf{G}_n^M \coloneqq \{\mathsf{\beta} \in \mathsf{G}_n : \|\mathsf{\beta} - \beta\| \le n^{-1/2}M\}$. For fixed M, the

[39] That is, $\bar{\beta}_n$ is the nearest element in G_n to $\hat{\beta}_n$.

cardinality $|\mathsf{G}_n^M| < \infty$ of this set is bounded independently of n, say by G^M. For any $\upsilon > 0$,

$$\begin{aligned} P_\theta^n\left(\|R_{n,i}(\bar{\beta}_n)\| > \upsilon\right) &\leq \varepsilon + \sum_{\beta_n^\star \in \mathsf{G}_n^M} \left(\{\|R_{n,i}(\beta_n^\star)\| > \upsilon\} \cap \bar{\beta}_n = \beta_n^\star\right) \\ &\leq \varepsilon + \sum_{\beta_n^\star \in \mathsf{G}_n^M} (\|R_{n,i}(\beta_n^\star)\| > \upsilon) \\ &\leq \varepsilon + \mathsf{G}^M P_\theta^n\left(\|R_{n,i}(\beta_n^\diamond)\| > \upsilon\right), \end{aligned}$$

where $\breve{\beta}_n \in \mathsf{G}_n^M$ is the maximiser of $\beta^\star \mapsto P_\theta^n(\|R_{n,i}(\beta^\star)\| > \upsilon)$. As $\breve{\beta}_n \in \mathsf{G}_n^M$, $\breve{\theta}_n \coloneqq (\alpha, \breve{\beta}_n, \eta)$ is a deterministic sequence with $\sqrt{n}\|\breve{\beta}_n - \beta\| = O_{P_\theta^n}(1)$. Thus, by equation (37) and Step 1,

$$\begin{aligned} \sqrt{n}\mathbb{P}_n\left[\tilde{\ell}_{\bar{\theta}_n} - \tilde{\ell}_\theta\right] + \sqrt{n}\tilde{I}_\theta \begin{pmatrix} 0 \\ \bar{\beta}_n - \beta \end{pmatrix} &= o_{P_\theta^n}(1); \\ \sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\bar{\gamma}_n} - \tilde{\ell}_{\bar{\theta}_n}\right] &= o_{P_\theta^n}(1); \\ \|\hat{I}_{n,\bar{\gamma}_n} - \tilde{I}_\theta\|_2 &= o_{P_\theta^n}(\nu_n^{1/2}). \end{aligned} \tag{41}$$

Step 3: Combine the first two lines of (41) to obtain

$$\sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\bar{\gamma}_n} - \tilde{\ell}_\theta\right] = -\sqrt{n}\tilde{I}_\theta \begin{pmatrix} 0 \\ \bar{\beta}_n - \beta \end{pmatrix} + o_{P_\theta^n}(1).$$

By the third line of (41),

$$\hat{\mathcal{K}}_{n,\bar{\gamma}_n} \coloneqq \left[I \quad -\hat{I}_{n,\bar{\gamma}_n,\alpha\beta}\hat{I}_{n,\bar{\gamma}_n,\beta\beta}^{-1}\right] \xrightarrow{P_\theta^n} \tilde{\mathcal{K}}_\theta \coloneqq \left[I \quad -\tilde{I}_{\theta,\alpha\beta}\tilde{I}_{\theta,\beta\beta}^{-1}\right].$$

By (38) and Example 6.5 in van der Vaart (1998), $P_{\theta_n(g_n,h_n)}^n \lhd\rhd P_\theta^n$. In combination with the preceding two displays this gives

$$\begin{aligned} &\sqrt{n}\mathbb{P}_n\left[\hat{\kappa}_{n,\bar{\gamma}_n} - \tilde{\kappa}_\theta\right] \\ &\quad = \left[\hat{\mathcal{K}}_{n,\bar{\gamma}_n} - \tilde{\mathcal{K}}_\theta\right]\sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\bar{\gamma}_n} - \tilde{\ell}_\theta\right] + \tilde{\mathcal{K}}_\theta\sqrt{n}\mathbb{P}_n\left[\hat{\ell}_{n,\bar{\gamma}_n} - \tilde{\ell}_\theta\right] + \left[\hat{\mathcal{K}}_{n,\bar{\gamma}_n} - \tilde{\mathcal{K}}_\theta\right]\sqrt{n}\mathbb{P}_n\tilde{\ell}_\theta \\ &\quad = -\tilde{\mathcal{K}}_\theta\tilde{I}_\theta \begin{pmatrix} 0 \\ \sqrt{n}(\bar{\beta}_n - \beta) \end{pmatrix} + o_{P_{\theta_n(g_n,h_n)}^n}(1) \\ &\quad = -\left(\tilde{\mathcal{I}}_\theta \quad 0\right) \begin{pmatrix} 0 \\ \sqrt{n}(\bar{\beta}_n - \beta) \end{pmatrix} + o_{P_{\theta_n(g_n,h_n)}^n}(1) \\ &\quad = o_{P_{\theta_n(g_n,h_n)}^n}(1). \end{aligned}$$

By (38) and Le Cam's third Lemma (e.g. van der Vaart, 1998, Example 6.7),

$$\sqrt{n}\mathbb{P}_n\tilde{\kappa}_\theta = \mathcal{K}_\theta\sqrt{n}\mathbb{P}_n\tilde{\ell}_\theta \rightsquigarrow \mathcal{K}_\theta\mathcal{Z}, \qquad \text{where } \mathcal{Z} \sim \mathcal{N}(\tilde{I}_\theta g, \tilde{I}_\theta)$$

under any $P^n_{\theta_n(g_n,h_n)}$ with $(g_n,h_n)\to(g,h)$ (all in $\mathcal{V}$). $\mathcal{K}_\theta\tilde{I}_\theta\mathcal{K}_\theta' = \tilde{\mathcal{I}}$ and with $g=(0,(b,s)')'$,

$$\mathcal{K}_\theta\tilde{I}_\theta g = \begin{pmatrix}\tilde{\mathcal{I}}_\theta & 0\end{pmatrix}\begin{pmatrix}0\\(b,s)'\end{pmatrix} = 0.$$

We conclude that

$$\hat{\kappa}_{n,\bar{\gamma}_n} \rightsquigarrow \mathcal{N}(0,\tilde{\mathcal{I}}_\theta) \quad \text{under } P^n_{\theta_n(g_n,h_n)}. \tag{42}$$

For the final part of the proof, note that since any submatrix has a smaller operator norm than the original matrix and the matrix inverse is Lipschitz continuous at a non-singular matrix, the third line of (41) implies that

$$\|\hat{\mathcal{I}}_{n,\bar{\gamma}_n} - \tilde{\mathcal{I}}_\theta\|_2 = o_{P^n_\theta}(\nu_n^{1/2}).$$

Therefore, by Proposition S1 and $P^n_{\theta_n(g_n,h_n)} \triangleleft\triangleright P^n_\theta$,

$$\hat{\mathcal{I}}^{t,\dagger}_{n,\bar{\gamma}_n} \xrightarrow{P^n_{\theta_n(g_n,h_n)}} \tilde{\mathcal{I}}^\dagger_\theta \quad \text{and} \quad \hat{r}_{n,\bar{\gamma}_n} \xrightarrow{P^n_{\theta_n(g_n,h_n)}} r,$$

which gives (40). For the final part of (39), combine the preceding display with the weak convergence result in equation (42) and Theorem 9.2.2 in Rao and Mitra (1971). □

Corollary 1. *In the setting of Theorem 2, let c_n be the $1-a$ quantile of the $\chi^2_{r_n}$ distribution for any $a\in(0,1)$ and*

$$\Theta_{0,n} = \left\{(\alpha_0, \beta + d/\sqrt{n}, \eta(1+h/\sqrt{n}) : d\in D^\star, h\in H^\star\right\},$$

where $D^\star$ is a bounded subset of $\mathbb{R}^{L_\beta}$ and $H^\star$ is a compact subset of H.[40] *Then,*

$$\lim_{n\to\infty}\sup_{\vartheta\in\Theta_{0,n}} P^n_\vartheta\left(\hat{S}_{n,\bar{\gamma}_n} > c_n\right) \le a,$$

with inequality only if $r=0$.

Proof. Set $\hat{S}_n := \hat{S}_{n,\bar{\gamma}_n}$, $\hat{r}_n := \hat{r}_{n,\bar{\gamma}_n}$ and $\varphi_n := \mathbf{1}\{\hat{S}_n > c_n\}$. Let g,h be such that $g=(0,d)$, $d\in D^\star$ and $h\in H^\star$. Since $\hat{r}_n \xrightarrow{P^n_\theta} r$ (by Theorem 2), the events $E_n := \{\hat{r}_n = r\}$ satisfy $P^n_\theta E_n \to 1$. Thus $c_n \xrightarrow{P^n_\theta} c$, the $1-a$ quantile of a χ^2_r random variable. We now split into cases.

Case 1: $r>0$. By Theorem 2

$$\hat{S}_n - c_n \rightsquigarrow \mathcal{Z} - c \text{ under } P^n_{\theta_n(g,h)} \text{ as } n\to\infty,$$

with $\mathcal{Z}\sim\chi^2_r$. Since this is a continuous distribution

$$\lim_{n\to\infty} P^n_{\theta_n(g,h)}\varphi_n = a.$$

Case 2: $r=0$. On E_n, $\hat{r}_n = 0 \implies \hat{\mathcal{I}}_{n,\bar{\gamma}_n} = 0 \implies \hat{S}_n = 0 \implies \varphi_n = 0$, whilst

[40] See the discussion immediately preceding Lemma 2 for the norm used on H.

$P^n_{\theta_n(g,h)} E_n \to 1$ by the contiguity which follows from (38) and Example 6.5 in van der Vaart (1998). Thus

$$\lim_{n\to\infty} P^n_{\theta_n(g,h)} \varphi_n = 0.$$

These two limiting statements continue to hold under any convergent sequence $(g_n, h_n) \to (g,h)$, with each $g_n = (0, d_n)$ for $d_n \in D^\star$ and $h_n \in H^\star$ and $(g,h) \in \operatorname{cl} D^\star \times H^\star$, as follows directly from $d_{TV}(P^n_{\theta_n(g_n,h_n)}, P^n_{\theta_n(g,h)}) \to 0$ as shown in Lemma S7. Considering such convergent sequences is sufficient since each $(g_n, h_n) \in \{0\} \times \operatorname{cl} D^\star \times H^\star$, which is compact. □

We next prove our main Theorem by verifying the conditions of Corollary 1.

Proof of Theorem 1. It suffices to show the conditions of Corollary 1 hold. There are 4 conditions which we verify in order: items 1, 2, 3 & equation (38) of the statement of Theorem 2.

Condition 1: Let $d_n := \sqrt{n}(\beta_n - \beta)$ and $g_n = (0, d_n)$. Then $\tilde{\theta}_n = \theta_n(g_n, 0)$. By arguing along subsequences if necessary we may assume without loss of generality that $d_n \to d$. By Theorem 12.14 in Rudin (1991),

$$P_\theta \left[\tilde{\ell}_\theta \dot{\ell}_\theta'\right] g = \tilde{I}_\theta \begin{pmatrix} 0 \\ d \end{pmatrix} = \tilde{I}_\theta \begin{pmatrix} 0 \\ \sqrt{n}(\beta_n - b) \end{pmatrix} + o(1). \tag{43}$$

Given this, condition 1 follows by Proposition A.10 in van der Vaart (1988), the hypotheses of which are verified by Lemmas 1, S9 and S10.

Condition 2: This follows by repeated addition and subtraction along with the convergence in probability and stochastic boundedness results of Lemma S11, Lemma 4, the moment conditions in Assumption 2 and the boundedness of $A(\alpha, \sigma_n)$, $A(\alpha, \sigma_n)^{-1}$ and $D_{x,l}(\alpha, \sigma_n)$ (for $x \in \{\alpha, \sigma\}$), which follows as each of these functions is continuous by Assumption 1 and $(\sigma_n)_{n\in\mathbb{N}}$ is a convergent sequence.

Condition 3: Let $\breve{I}_{n,\theta_n} := \frac{1}{n}\sum_{i=1}^n \tilde{\ell}_{\tilde{\theta}_n} \tilde{\ell}_{\tilde{\theta}_n}'$. By repeated addition and subtraction along with the results of Lemmas 4, S13 and S14,

$$\frac{1}{n}\sum_{i=1}^n \|\tilde{\ell}_{\tilde{\theta}_n} - \hat{\ell}_{n,\gamma_n}\|^2 = o_{P^n_{\tilde{\theta}_n}}(\nu_n).$$

This and Lemma S9 imply that

$$\begin{aligned}
\left\|\hat{I}_{n,\gamma_n} - \breve{I}_{n,\tilde{\theta}_n}\right\|_2 &= \left\|\frac{1}{n}\sum_{i=1}^{n} \hat{\ell}_{n,\gamma_n}\left(\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right)' + \left(\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right)\tilde{\ell}'_{\tilde{\theta}_n}\right\|_2 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left\|\hat{\ell}_{n,\gamma_n}\left(\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right)'\right\|_2 + \frac{1}{n}\sum_{i=1}^{n}\left\|\left(\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right)\tilde{\ell}'_{\tilde{\theta}_n}\right\|_2 \\
&\leq \left(\frac{1}{n}\sum_{i=1}^{n}\left\|\hat{\ell}_{n,\gamma_n}\right\|^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right\|^2\right)^{1/2} \\
&\qquad + \left(\frac{1}{n}\sum_{i=1}^{n}\left\|\hat{\ell}_{n,\gamma_n} - \tilde{\ell}_{\tilde{\theta}_n}\right\|^2\right)^{1/2}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\tilde{\ell}_{\tilde{\theta}_n}\right\|^2\right)^{1/2} \\
&= o_{P^n_{\tilde{\theta}_n}}(\nu_n^{1/2}).
\end{aligned}$$

To complete the demonstration of Condition 3, we show that the right hand side terms in

$$\|\breve{I}_{n,\tilde{\theta}_n} - \tilde{I}_\theta\| \leq \left\|\mathbb{P}_n\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n} - P_{\tilde{\theta}_n}\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n}\right]\right]\right\| + \left\|P_{\tilde{\theta}_n}\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n}\right] - P_\theta\left[\tilde{\ell}_\theta\tilde{\ell}'_\theta\right]\right\|$$

are respectively $o_{P^n_{\tilde{\theta}_n}}(\nu_n^{1/2})$ and $o(\nu_n^{1/2})$. Under $P^n_{\tilde{\theta}_n}$, each $e'_k A(\alpha, \sigma_n)V_{\tilde{\theta}_n,i}$ has the same law as $\epsilon_{k,i}$ $(k = 1, \ldots, K)$, whilst the same is true for $A(\alpha,\sigma)V_{\theta,i}$ under P^n_θ. This, $\sqrt{n}\|\beta_n - \beta\| = O(1)$ and the local Lipschitz continuity of each $\beta \mapsto \zeta^x_{l,j,k}(\alpha,\sigma)$ and $\beta \mapsto A(\alpha,\sigma)$ yield that the rightmost term is $O(n^{-1/2}) = o(\nu_n^{1/2})$. For the first term on the right hand side we note that $\sup_{n\in\mathbb{N}} P_{\tilde{\theta}_n}\|\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n}\|^{2+\delta/2} < \infty$ by Lemma S9. This is sufficient as either $1 + \delta/4 > p = 2$, in which case $\mathbb{P}_n\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n} - P_{\tilde{\theta}_n}\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n}\right]\right] = O_{P^n_{\tilde{\theta}_n}}(n^{-1/2}) = o_{P^n_{\tilde{\theta}_n}}(\nu_n^{1/2})$ by Lindeberg's CLT or $p = 1 + \delta/4 \in (1,2)$ whence $\mathbb{P}_n\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n} - P_{\tilde{\theta}_n}\left[\tilde{\ell}_{\tilde{\theta}_n}\tilde{\ell}'_{\tilde{\theta}_n}\right]\right] = O_{P^n_{\tilde{\theta}_n}}(n^{(1-p)/p}) = o_{P^n_{\tilde{\theta}_n}}(\nu_n^{1/2})$ by a Marcinkiewicz – Zygmund style weak law of large numbers for triangular arrays.[41]

Condition 4: By Lemma 1, Lemma 1.7 of van der Vaart (2002) and Theorem I.2.7 of Conway (1985), the random vector

$$\left(\tilde{\ell}_\theta(W_i),\ g'\dot{\ell}_\theta(W_i) + \sum_{k=0}^{K}\tilde{h}_k(W_i)\right)$$

is zero mean and has a finite variance matrix under P_θ. By the definition of $\tilde{\ell}_\theta$ as an

[41] A formal statemement is as follows: Let $(X_{n,i})_{n\in\mathbb{N},1\leq i\leq n}$ be a triangular array of zero-mean random variables, i.i.d. along rows. Let $S_n \coloneqq \sum_{i=1}^n X_{n,i}$. If $\sup_{n\in\mathbb{N}}\mathbb{E}|X_{n,1}|^p < \infty$ for $p \in (1,2)$, then $S_n/n^{1/p}$ converges to zero in probability as $n \to \infty$. For the case of an i.i.d. sequence (in place of a triangular array) this result is recorded as, for example, Theorem 6.3.2 of Gut (2005); the proof given there extends essentially verbatim to the case considered here.

orthogonal projection and Theorem 12.14 in Rudin (1991), one has

$$P_\theta\left[\tilde{\ell}_\theta\left(g'\dot{\ell}_\theta+\sum_{k=0}^{K}\tilde{h}_k\right)\right]=P_\theta\left[\tilde{\ell}_\theta\dot{\ell}_\theta'\right]g=\tilde{I}_\theta g.$$

Therefore, by the central limit theorem, under P_θ^n

$$\sqrt{n}\mathbb{P}_n\left(\tilde{\ell}_\theta,\ g'\dot{\ell}_\theta+\sum_{k=0}^{K}\tilde{h}_k\right)\rightsquigarrow\mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\tilde{I}_\theta & \tilde{I}_\theta g\\ g'\tilde{I}_\theta & \sigma(g,h)\end{pmatrix}\right), \tag{44}$$

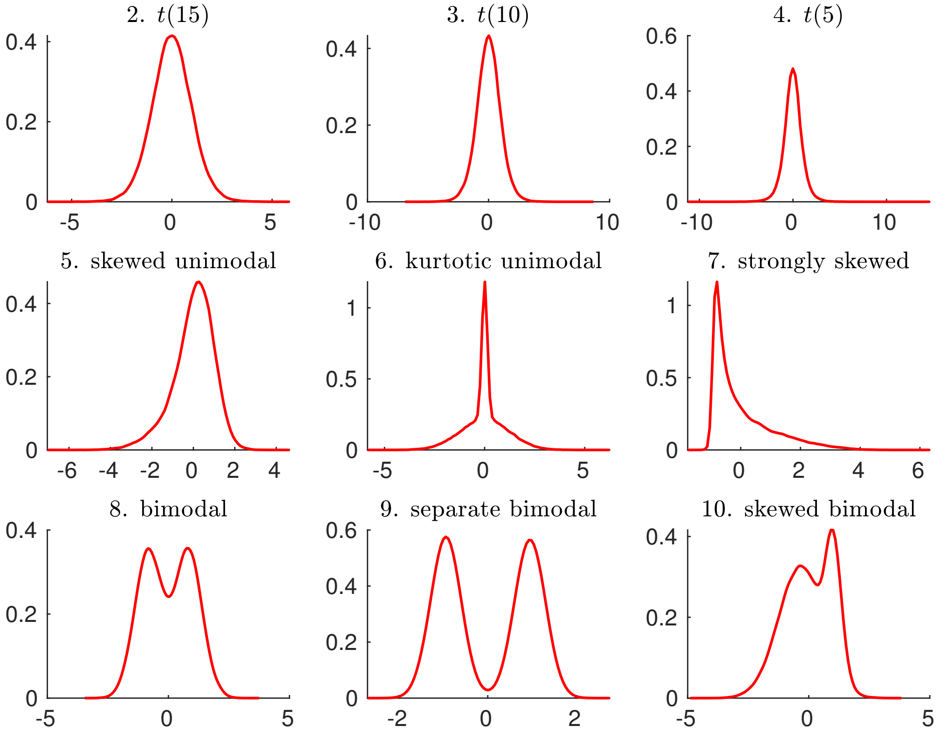
where

$$\sigma(g,h)\coloneqq P_\theta\left[g'\dot{\ell}_\theta+\sum_{k=0}^{K}\tilde{h}_k\right]^2.$$

Combination of this with Lemma 2 and equation (30) verifies (38). □

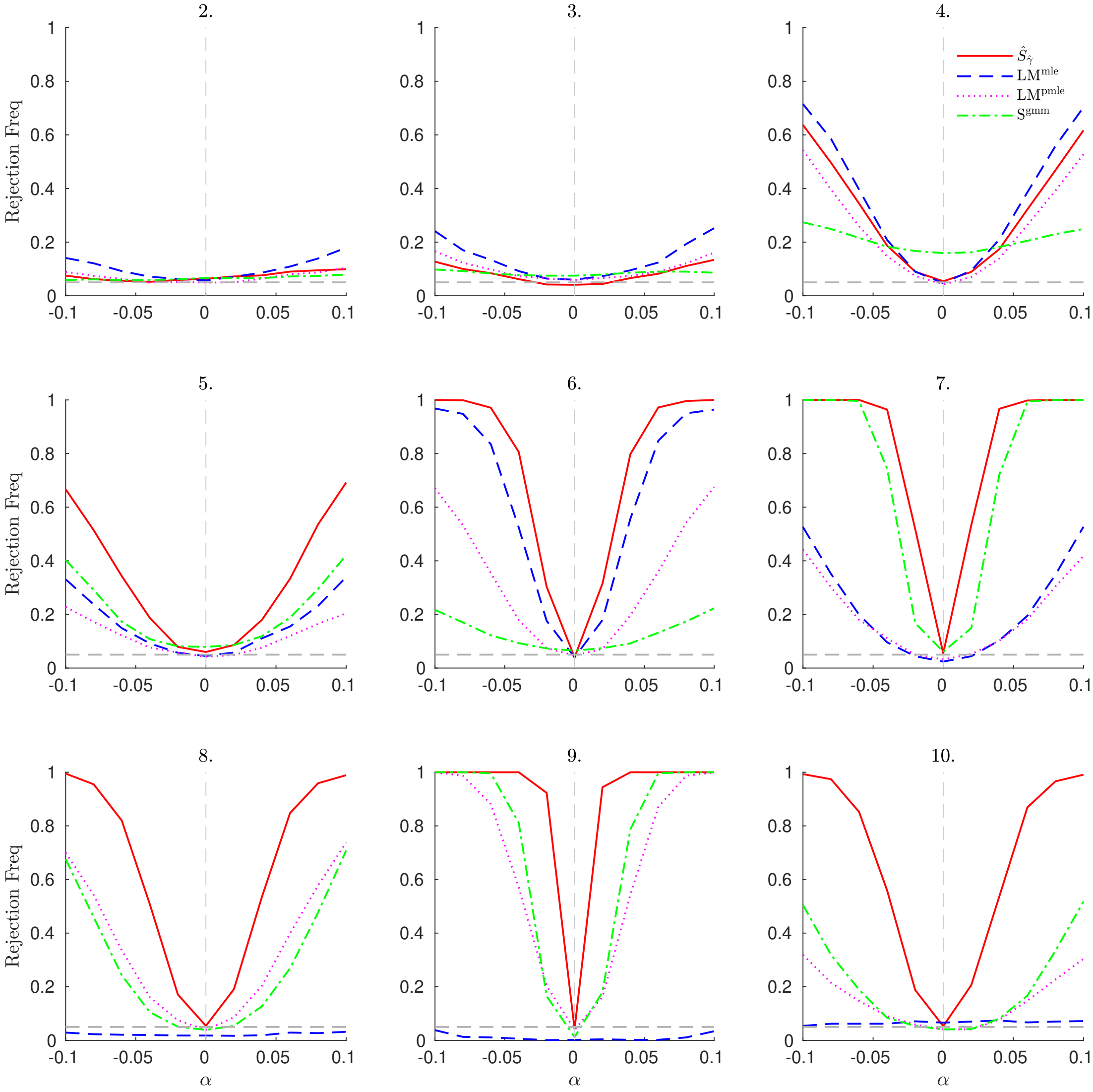
D Figures and tables

Figure 3: Structural Shock Densities



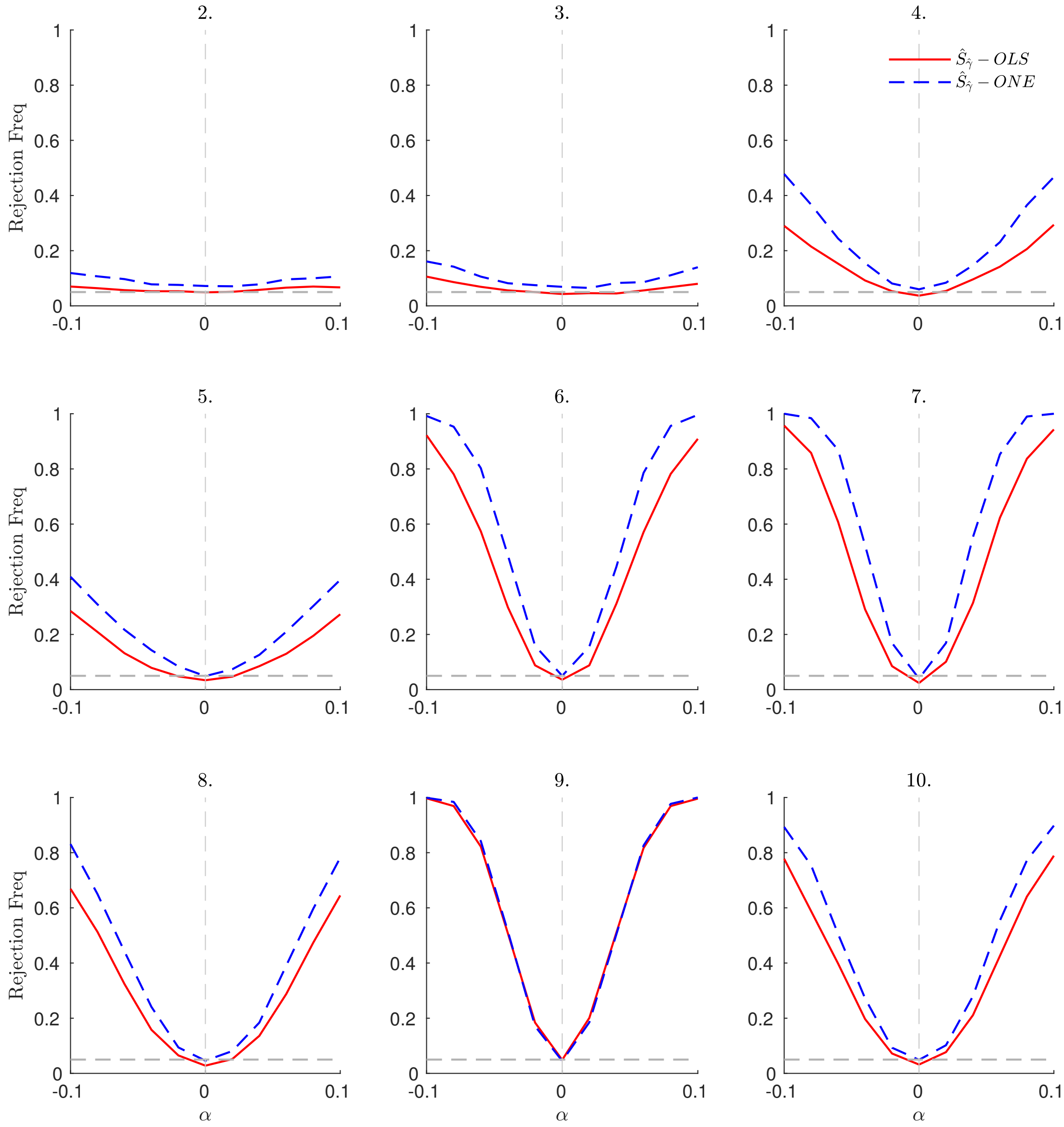
Notes: The plots show the different densities considered for simulating the structural shocks. Densities 2-4 are t-distributions normalised to have unit variance. Densities 5 - 10 (and their names) are mixtures of normals taken from Marron and Wand (1992); see their table 1 for the definitions. Density 1 is the standard Gaussian and omitted from the figure.

Figure 4: Power Comparison Baseline model



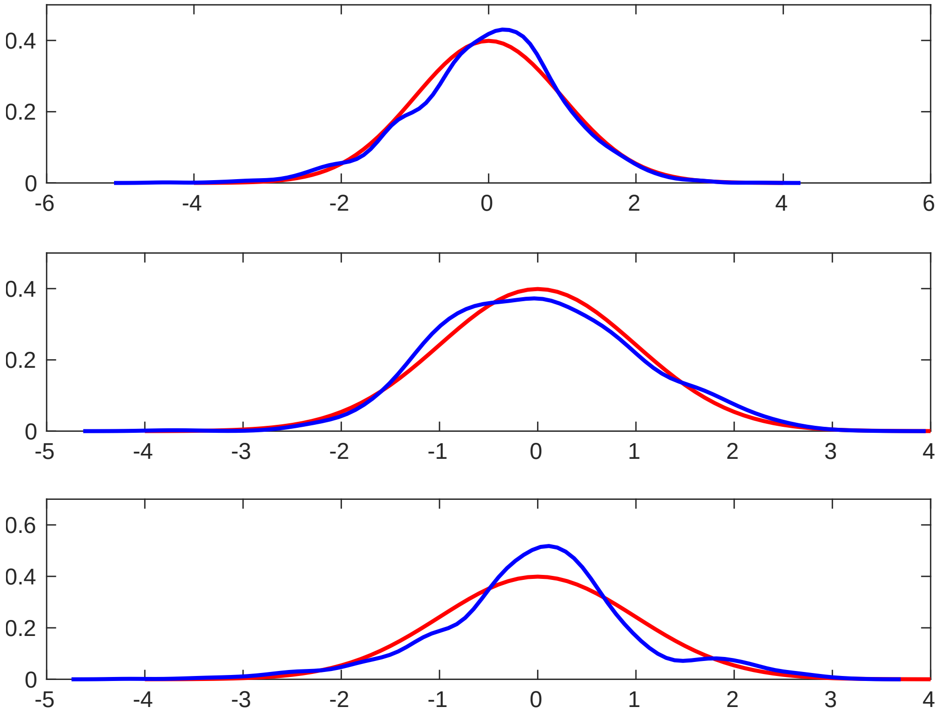
Notes: Empirical power curves for the baseline model with $k = 2$ and $n = 1000$. Each plot corresponds to the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3. The solid red line corresponds to $S_{\hat{\gamma}}$, the dashed blue line to LM$^{\text{mle}}$, the dotted pink line to LM$^{\text{pmle}}$ and the dot-dashed green line to S$^{\text{gmm}}$.

Figure 5: Power LSEM



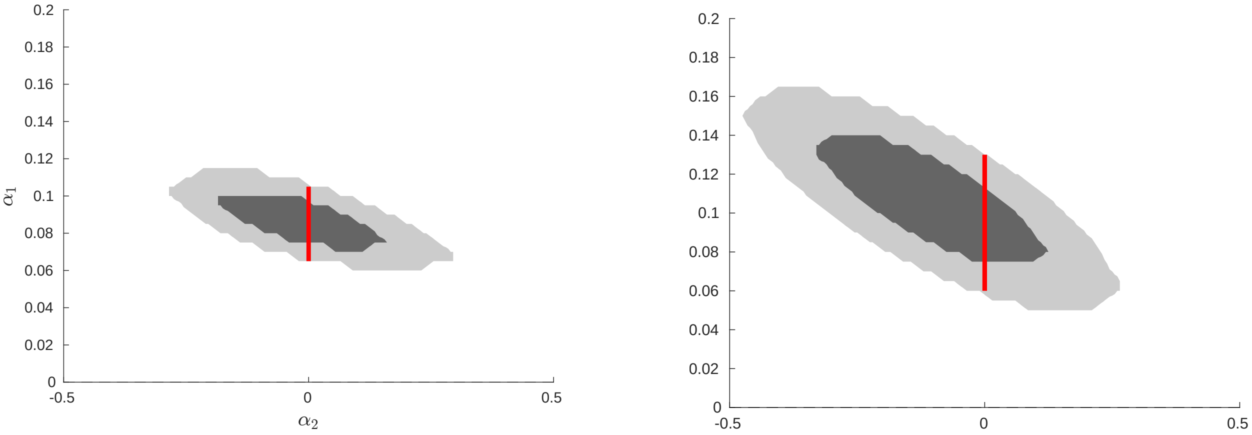
Notes: Empirical power curves for the LSEM model with $k = 2$, $d = 2$ and $n = 1000$. Each plot corresponds to the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3. The solid red line corresponds to the empirical rejection frequency of the $\hat{S}_{\hat{\gamma}}$ test where $\hat{\gamma} = (\alpha_0, \hat{\beta})$, with $\hat{\beta}$ the OLS estimator. The dashed blue line corresponds to the empirical rejection frequency of the $\hat{S}_{\hat{\gamma}}$ test where $\hat{\gamma} = (\alpha_0, \hat{\beta})$, with $\hat{\beta}$ the one-step efficient MLE estimator.

Figure 6: Densities: returns to schooling



Notes: We show the kernel density estimates for $\hat{\epsilon}_{i,1}$, $\hat{\epsilon}_{i,2}$ and $\hat{\epsilon}_{i,3}$ (blue line) together with the pdf of the standard normal distribution (red line). The error estimates are obtained as $\hat{\epsilon}_i = \hat{A}\hat{V}_i$, where $\hat{A} = A(\tilde{\alpha}_1, \hat{\sigma})$ with $\tilde{\alpha}_1$ being the value that minimizes the score statistic.

Figure 7: Confidence sets: returns to schooling



(a) Semi-parametric score test

(b) Pseudo MLE LM test

Notes: We show 95% (light gray) and 67% (dark gray) confidence sets for $\alpha = (\alpha_1, \alpha_2)$, where α_1 captures the effect of education on log wages and α_2 capture the correlation between the instrument (proximity to schooling interacted with parental education) and the error of the log wage equation. The red line indicates the confidence interval under the restriction of instrument exogeneity, i.e. $\alpha_2 = 0$. Figure (a) shows the result after inverting the weak non-Gaussianity robust test $\hat{S}_{\hat{\gamma}}$. Figure (b) shows the result after inverting the pseudo MLE LM test based on the Student's t density.

Table 2: Rejection Frequencies $\hat{S}_{\hat{\gamma}}$ test for Baseline model

n	K	B	1	2	3	4	5	6	7	8	9	10
200	2	4	0.049	0.049	0.048	0.040	0.047	0.049	0.034	0.049	0.048	0.048
200	2	6	0.048	0.045	0.049	0.044	0.048	0.053	0.047	0.045	0.058	0.051
200	2	8	0.050	0.049	0.047	0.044	0.048	0.048	0.053	0.050	0.051	0.047
200	3	4	0.043	0.039	0.039	0.039	0.044	0.048	0.026	0.049	0.052	0.050
200	3	6	0.045	0.038	0.040	0.044	0.041	0.048	0.044	0.047	0.052	0.043
200	3	8	0.047	0.046	0.040	0.040	0.044	0.048	0.042	0.049	0.044	0.051
200	5	4	0.032	0.034	0.033	0.034	0.035	0.039	0.015	0.041	0.045	0.043
200	5	6	0.037	0.033	0.036	0.032	0.032	0.040	0.043	0.045	0.043	0.044
200	5	8	0.039	0.038	0.038	0.030	0.035	0.043	0.045	0.040	0.041	0.038
500	2	4	0.053	0.046	0.053	0.045	0.047	0.052	0.031	0.049	0.045	0.046
500	2	6	0.048	0.049	0.048	0.048	0.049	0.052	0.057	0.047	0.047	0.049
500	2	8	0.048	0.048	0.045	0.049	0.047	0.045	0.051	0.052	0.048	0.045
500	3	4	0.042	0.039	0.040	0.046	0.048	0.048	0.021	0.042	0.046	0.047
500	3	6	0.043	0.045	0.042	0.042	0.045	0.047	0.047	0.051	0.044	0.045
500	3	8	0.046	0.045	0.040	0.035	0.042	0.047	0.044	0.045	0.050	0.047
500	5	4	0.040	0.036	0.039	0.036	0.041	0.046	0.016	0.048	0.047	0.046
500	5	6	0.041	0.039	0.039	0.039	0.040	0.049	0.046	0.045	0.044	0.044
500	5	8	0.039	0.040	0.036	0.041	0.043	0.050	0.050	0.044	0.046	0.047
1000	2	4	0.042	0.052	0.040	0.055	0.047	0.052	0.046	0.052	0.046	0.048
1000	2	6	0.054	0.052	0.045	0.050	0.045	0.049	0.049	0.054	0.045	0.057
1000	2	8	0.047	0.048	0.048	0.047	0.048	0.052	0.050	0.048	0.055	0.052
1000	3	4	0.049	0.041	0.043	0.045	0.048	0.050	0.054	0.051	0.051	0.047
1000	3	6	0.048	0.044	0.038	0.040	0.050	0.047	0.046	0.049	0.051	0.045
1000	3	8	0.046	0.047	0.047	0.042	0.049	0.045	0.050	0.052	0.043	0.047
1000	5	4	0.038	0.035	0.038	0.047	0.041	0.044	0.050	0.046	0.047	0.048
1000	5	6	0.041	0.043	0.039	0.042	0.043	0.049	0.044	0.048	0.048	0.049
1000	5	8	0.042	0.042	0.038	0.039	0.048	0.050	0.049	0.047	0.045	0.049

Notes: The table shows the empirical rejection frequencies for the $S_{\hat{\gamma}}$ test based on $S = 5,000$ Monte Carlo replications for the baseline model $Y_i = A^{-1}\epsilon_i$. The test has nominal level $a = 0.05$. The columns denote the sample size n, the dimension of the model K, the number of B-splines B and the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3.

Table 3: Rejection Frequencies Alternative Tests for Baseline model

Cat (i)	n	1	2	3	4	5	6	7	8	9	10
$\mathrm{W}^{\mathrm{mle}}$	200	0.179	0.149	0.139	0.127	0.113	0.059	0.097	0.152	0.125	0.171
	500	0.180	0.133	0.114	0.115	0.095	0.167	0.073	0.114	0.097	0.150
	1000	0.188	0.101	0.079	0.074	0.061	0.405	0.058	0.124	0.103	0.170
$\mathrm{LR}^{\mathrm{mle}}$	200	0.028	0.054	0.060	0.046	0.054	0.026	0.048	0.017	0.018	0.024
	500	0.043	0.056	0.068	0.054	0.065	0.023	0.053	0.016	0.017	0.024
	1000	0.049	0.065	0.063	0.061	0.053	0.031	0.051	0.022	0.018	0.025
$\mathrm{W}^{\mathrm{pmle}}$	200	0.375	0.211	0.198	0.086	0.141	0.058	0.105	0.495	0.998	0.467
	500	0.485	0.264	0.204	0.073	0.163	0.030	0.079	0.973	0.999	0.870
	1000	0.570	0.230	0.180	0.051	0.131	0.023	0.068	0.428	1.000	0.947
$\mathrm{LR}^{\mathrm{gmm}}$	200	0.413	0.411	0.425	0.441	0.290	0.379	0.120	0.216	0.086	0.232
	500	0.292	0.246	0.246	0.286	0.141	0.171	0.025	0.109	0.066	0.106
	1000	0.232	0.181	0.155	0.176	0.074	0.115	0.014	0.068	0.059	0.049
Cat (ii)	n	1	2	3	4	5	6	7	8	9	10
$\hat{S}_{\hat{\gamma}}$	200	0.051	0.047	0.048	0.040	0.049	0.049	0.047	0.048	0.050	0.044
	500	0.047	0.047	0.054	0.047	0.044	0.043	0.047	0.048	0.051	0.054
	1000	0.047	0.043	0.046	0.049	0.048	0.047	0.050	0.044	0.049	0.043
$\mathrm{LM}^{\mathrm{mle}}$	200	0.052	0.058	0.054	0.043	0.040	0.043	0.023	0.018	0.002	0.059
	500	0.056	0.052	0.052	0.042	0.046	0.047	0.028	0.017	0.001	0.062
	1000	0.062	0.052	0.050	0.049	0.039	0.040	0.029	0.016	0.002	0.052
$\mathrm{LM}^{\mathrm{plme}}$	200	0.049	0.045	0.049	0.035	0.038	0.046	0.030	0.041	0.042	0.042
	500	0.049	0.047	0.050	0.039	0.047	0.046	0.034	0.046	0.044	0.051
	1000	0.046	0.048	0.053	0.044	0.041	0.046	0.034	0.042	0.052	0.047
$\mathrm{S}^{\mathrm{gmm}}$	200	0.188	0.209	0.248	0.326	0.236	0.264	0.195	0.108	0.059	0.130
	500	0.094	0.105	0.123	0.223	0.116	0.133	0.103	0.057	0.028	0.064
	1000	0.061	0.070	0.081	0.162	0.069	0.078	0.054	0.031	0.019	0.035

Notes: The table shows the empirical rejection frequencies based on $S = 5,000$ Monte Carlo replications for the baseline model $Y_i = A^{-1}\epsilon_i$, with $n = 500$ and $K = 2$. All tests have nominal level $a = 0.05$. The first column indicates the test the second the sample size. The remaining columns denote the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3.

Table 4: Rejection Frequencies $\hat{S}_{\hat{\gamma}}$ test for LSEM - OLS $\hat{\beta}$

n	K	d	1	2	3	4	5	6	7	8	9	10
200	2	2	0.050	0.054	0.049	0.049	0.038	0.030	0.038	0.043	0.057	0.046
200	2	3	0.049	0.054	0.054	0.048	0.046	0.059	0.042	0.035	0.029	0.052
200	3	2	0.056	0.058	0.050	0.062	0.059	0.031	0.018	0.038	0.047	0.050
200	3	3	0.063	0.054	0.057	0.065	0.060	0.025	0.023	0.051	0.058	0.049
200	5	2	0.098	0.104	0.109	0.142	0.094	0.051	0.064	0.054	0.023	0.057
200	5	3	0.116	0.116	0.131	0.155	0.103	0.039	0.029	0.061	0.026	0.072
500	2	2	0.049	0.050	0.039	0.042	0.041	0.027	0.029	0.036	0.026	0.029
500	2	3	0.048	0.041	0.047	0.047	0.037	0.029	0.024	0.034	0.050	0.051
500	3	2	0.051	0.051	0.048	0.040	0.037	0.028	0.029	0.038	0.022	0.039
500	3	3	0.048	0.050	0.047	0.051	0.053	0.028	0.048	0.041	0.037	0.036
500	5	2	0.071	0.078	0.068	0.081	0.049	0.023	0.060	0.042	0.039	0.038
500	5	3	0.067	0.068	0.080	0.085	0.063	0.022	0.045	0.049	0.027	0.051
1000	2	2	0.040	0.051	0.049	0.029	0.043	0.032	0.033	0.045	0.049	0.041
1000	2	3	0.048	0.044	0.040	0.040	0.040	0.030	0.038	0.046	0.030	0.044
1000	3	2	0.045	0.038	0.043	0.034	0.033	0.032	0.034	0.040	0.039	0.042
1000	3	3	0.044	0.045	0.043	0.036	0.030	0.032	0.035	0.040	0.024	0.034
1000	5	2	0.059	0.051	0.057	0.051	0.039	0.024	0.063	0.030	0.028	0.036
1000	5	3	0.057	0.058	0.056	0.050	0.035	0.018	0.046	0.036	0.029	0.040

Notes: The table shows the empirical rejection frequencies for the $S_{\hat{\gamma}}$ test based on $S = 5,000$ Monte Carlo replications for the linear simultaneous equations model. The test has nominal level $a = 0.05$. The columns denote the sample size n, the dimension of the model K, the number of covariates d and the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3. The $S_{\hat{\gamma}}$ test was implemented using $B = 6$ B-splines.

Table 5: Rejection Frequencies $\hat{S}_{\hat{\gamma}}$ test for LSEM - One-step $\hat{\beta}$

n	K	d	1	2	3	4	5	6	7	8	9	10
200	2	2	0.067	0.080	0.068	0.081	0.070	0.031	0.054	0.056	0.061	0.051
200	2	3	0.068	0.074	0.076	0.072	0.066	0.071	0.057	0.047	0.026	0.061
200	3	2	0.095	0.106	0.104	0.120	0.090	0.041	0.026	0.059	0.036	0.061
200	3	3	0.099	0.103	0.105	0.114	0.098	0.037	0.028	0.071	0.035	0.064
200	5	2	0.187	0.226	0.247	0.264	0.178	0.063	0.040	0.072	0.020	0.068
200	5	3	0.212	0.238	0.262	0.289	0.193	0.064	0.049	0.089	0.036	0.088
500	2	2	0.062	0.062	0.068	0.067	0.057	0.034	0.049	0.041	0.021	0.037
500	2	3	0.059	0.064	0.071	0.069	0.056	0.031	0.019	0.046	0.031	0.051
500	3	2	0.078	0.078	0.081	0.079	0.066	0.026	0.024	0.047	0.021	0.045
500	3	3	0.076	0.081	0.091	0.088	0.068	0.025	0.029	0.050	0.042	0.042
500	5	2	0.112	0.149	0.158	0.181	0.097	0.036	0.035	0.060	0.030	0.044
500	5	3	0.129	0.151	0.168	0.180	0.101	0.033	0.023	0.069	0.031	0.058
1000	2	2	0.059	0.059	0.065	0.048	0.049	0.025	0.021	0.055	0.050	0.038
1000	2	3	0.060	0.060	0.060	0.068	0.057	0.038	0.052	0.050	0.027	0.051
1000	3	2	0.061	0.067	0.068	0.065	0.053	0.023	0.048	0.047	0.023	0.045
1000	3	3	0.064	0.066	0.072	0.070	0.054	0.040	0.016	0.047	0.022	0.041
1000	5	2	0.091	0.105	0.108	0.111	0.069	0.032	0.026	0.042	0.029	0.043
1000	5	3	0.085	0.102	0.120	0.103	0.065	0.026	0.020	0.047	0.026	0.050

Notes: The table shows the empirical rejection frequencies for the $\hat{S}_{\hat{\gamma}}$ test based on $S = 5,000$ Monte Carlo replications for the linear simultaneous equations model (3). The test has nominal level $a = 0.05$. The columns denote the sample size n, the dimension of the observations K, the number of covariates d and the choice for densities ϵ_{ik}, for $k \geq 2$, where the numbers correspond to the different densities shown in Figure 3. The $S_{\hat{\gamma}}$ test was implemented using $B = 6$ B-splines and using OLS estimates for β.

Table 6: Confidence intervals: returns to schooling

Method	Estimate	Conf Interval	Length
$\hat{S}_{\hat{\gamma}}$	-	[0.068 , 0.105]	0.037
AR	-	[0.041 , 0.127]	0.086
OLS	0.076	[0.068 , 0.084]	0.016
2SLS	0.084	[0.040 , 0.127]	0.087

Notes: We report the 95% confidence bands for the effect of education on log wages using the proximity to college interacted with parental education as instrument. The sample size is $n = 2320$ and the model includes control variables for experience, race, smsa and region. The OLS and 2SLS confidence intervals are based on inverting the t-statistic under a normal limiting distribution. The confidence bands corresponding to the semi-parametric score test are based on the $\hat{S}_{\gamma}$ implemented using $B = 6$ B-splines and OLS estimates for $\hat{\beta}$. The AR confidence band is based on inverting the Anderson-Rubin statistic.